Neutrino Mixing

March 20, 2014

We will adopt the conventions $m, n = e, \mu, \tau$ and $i, j = 1, 2, 3$.

$$|\nu_m\rangle = \sum_i U^*_{mi}|\nu_i\rangle \quad \Rightarrow \quad |\nu_i\rangle = \sum_m U_{mi}|\nu_m\rangle \quad \Rightarrow \quad \langle\nu_i| = \sum_m \langle\nu_m|U^*_{mi}$$

where the second relation can be derived as follows

$$\sum_m U_{mj}|\nu_m\rangle = \sum_m\sum_i U_{mj}U^*_{mi}|\nu_i\rangle = \sum_m\sum_i U_{mj}U^\dagger_{im}|\nu_i\rangle = \sum_i \delta_{ij}|\nu_i\rangle = |\nu_j\rangle$$

From the expansion of fermion field $\psi(x)$, we see that

$$\psi(x)|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\sum_s v^s(p)e^{ip\cdot x}|\bar{\nu},\mathbf{p},s\rangle$$

Taking the charge conjugation transformation, we find

$$\mathsf{C}\psi(x)\mathsf{C}^{-1}|0\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\sum_s v^s(p)e^{ip\cdot x}|\nu,\mathbf{p},s\rangle$$

According to $\mathsf{C}\psi(x)\mathsf{C}^{-1} = \xi_c\gamma^2\psi^*(x)$, we can conclude that

$$|\bar{\nu}_n\rangle = \sum_j U_{nj}|\bar{\nu}_j\rangle \quad \Rightarrow \quad |\bar{\nu}_j\rangle = \sum_n U^*_{nj}|\bar{\nu}_n\rangle \quad \Rightarrow \quad \langle\bar{\nu}_j| = \sum_n \langle\bar{\nu}_n|U_{nj}$$

where the second relation can be derived as follows

$$\sum_n U^*_{ni}|\bar{\nu}_n\rangle = \sum_n\sum_j U^*_{ni}U_{nj}|\bar{\nu}_j\rangle = \sum_n\sum_j U^\dagger_{in}U_{nj}|\bar{\nu}_j\rangle = \sum_j \delta_{ij}|\bar{\nu}_j\rangle = |\bar{\nu}_i\rangle$$

With the above transformations, we proceed to consider the scattering process

$$\langle\nu_i\bar{\nu}_j|H_I|e^-e^+\rangle = \sum_m\sum_n \langle\nu_m\bar{\nu}_n|U^*_{mi}U_{nj}H_I|e^-e^+\rangle$$

Summing over the constraint $m = n$, we can obtain

$$\langle\nu_i\bar{\nu}_j|H_I|e^-e^+\rangle = \sum_m U^\dagger_{im}U_{mj}\langle\nu_m\bar{\nu}_m|H_I|e^-e^+\rangle$$

Let $\mathscr{M}_1$ denotes the contribution from the exchange of Z-boson and $\mathscr{M}_2$ denotes the contribution from the exchange of W-boson, it folows that

$$\mathscr{M}(e^-e^+ \to \nu_e\bar{\nu}_e) = \mathscr{M}_1 - \mathscr{M}_2, \quad \mathscr{M}(e^-e^+ \to \nu_\mu\bar{\nu}_\mu) = \mathscr{M}(e^-e^+ \to \nu_\tau\bar{\nu}_\tau) = \mathscr{M}_1$$

Using the relation $\sum_m U^\dagger_{im}U_{mj} = \delta_{ij}$, we can similarly get

$$\mathscr{M}(e^-e^+ \to \nu_i\bar{\nu}_j) = U^\dagger_{ie}U_{ej}(\mathscr{M}_1 - \mathscr{M}_2) + (U^\dagger_{i\mu}U_{\mu j} + U^\dagger_{i\tau}U_{\tau j})\mathscr{M}_1 = \delta_{ij}\mathscr{M}_1 - U^*_{ei}U_{ej}\mathscr{M}_2$$

Since $|\nu_i\rangle$ are mass eigenstates, their propagation can be described by plane wave solutions of the form

$$|\nu_i(t)\rangle = e^{-i(E_it - \vec{k}_i\cdot\vec{x})}|\nu_i(0)\rangle$$

In the ultrarelativistic limit $|\vec{k}_i| = k_i \gg m_i$, we can approximate the energy as

$$E_i = \sqrt{k_i^2 + m_i^2} \simeq k_i + \frac{m_i^2}{2k_i}$$

Using also $t \approx L$, where L is the distance traveled, the wavefunction becomes

$$|\nu_i(L)\rangle = e^{-im_i^2 L/2k_i}|\nu_i(0)\rangle \approx e^{-im_i^2 L/2E}|\nu_i(0)\rangle$$

where E is the energy of massless neutrino. Then the amplitude of the probability that neutrino ν_m will be observed can be written as

$$A(\nu_i \to \nu_m) = \langle \nu_m|\nu_i(t)\rangle = \sum_j U_{mj} e^{-im_i^2 L/2E}\langle \nu_j|\nu_i\rangle = U_{mi} e^{-im_i^2 L/2E}$$

For antineutrinos, we can obtain similarly

$$A(\bar{\nu}_j \to \bar{\nu}_n) = \langle \bar{\nu}_n|\bar{\nu}_j(t)\rangle = \sum_j U^*_{nj} e^{im_j^2 L/2E}\langle \bar{\nu}_i|\bar{\nu}_j\rangle = U^*_{nj} e^{-im_j^2 L/2E}$$

Then, we come to the point to express the amplitude of $e^- e^+ \to \nu_m \bar{\nu}_n$ as

$$\begin{aligned}
\mathscr{M}(e^- e^+ \to \nu_m \bar{\nu}_n) &= \sum_i \sum_j \mathscr{M}(e^- e^+ \to \nu_i \bar{\nu}_j) A(\upsilon_i \to \upsilon_m) A(\bar{\nu}_j \to \bar{\nu}_n) \\
&= \sum_i \sum_j (\delta_{ij}\mathscr{M}_1 - U^*_{ei} U_{ej}\mathscr{M}_2) U_{mi} U^*_{nj} e^{-i(m_i^2 - m_j^2)L/2E} \\
&= \sum_i U_{mi} U^*_{ni} - \sum_i \sum_j U^*_{ei} U_{ej} U_{mi} U^*_{nj} e^{-i(m_i^2 - m_j^2)L/2E}\mathscr{M}_2 \\
&= \delta_{mn}\mathscr{M}_1) - \sum_i \sum_j U^*_{ei} U_{ej} U_{mi} U^*_{nj} e^{-i\Delta m_{ij}^2 L/2E}\mathscr{M}_2
\end{aligned}$$

where $\Delta m_{ij}^2 = m_i^2 - m_j^2$. We proceed to calculate the square of the amplitude

$$\begin{aligned}
|\mathscr{M}(e^- e^+ \to \nu_m \bar{\nu}_n)|^2 = \delta_{mn}&\Big[|\mathscr{M}_1|^2 - 2\mathrm{Re}\Big(\sum_i \sum_j U^*_{ei} U_{ej} U_{mi} U^*_{nj} e^{-i\Delta m_{ij}^2 L/2E}\mathscr{M}_1^*\mathscr{M}_2\Big)\Big] \\
&+ \left|\sum_i \sum_j U^*_{ei} U_{ej} U_{mi} U^*_{nj} e^{-i\Delta m_{ij}^2 L/2E}\right|^2 |\mathscr{M}_2|^2
\end{aligned}$$

Taking $m = n$, we can obtain that

$$\begin{aligned}
\sum_i \sum_j U^*_{ei} U_{ej} U_{mi} U^*_{mj} e^{-i\Delta m_{ij}^2 L/2E} &= \sum_{i=j} (U^*_{ei} U_{mi})(U^*_{ej} U_{mj})^* + 2\mathrm{Re}\Big(\sum_{i>j} U^*_{ei} U_{ej} U_{mi} U^*_{mj} e^{-i\Delta m_{ij}^2 L/2E}\Big) \\
&= \sum_i |U^*_{ei} U_{mi}|^2 + 2\sum_{i>j} |U^*_{ei} U_{ej} U_{mi} U^*_{mj}| \cos\left(\frac{\Delta m_{ij}^2 L}{2E} - \varphi_{ij}\right)
\end{aligned}$$

where $\varphi_{ij} = \arg(U^*_{ei} U_{ej} U_{mi} U^*_{mj})$. If CP invariance holds, the above relation reduces to

$$\sum_i \sum_j U^*_{ei} U_{ej} U_{mi} U^*_{nj} e^{-i\Delta m_{ij}^2 L/2E} = \sum_i \sum_j U_{ei} U_{ej} U_{mi} U_{mj} \cos\left(\frac{\Delta m_{ij}^2 L}{2E}\right)$$

where the property $U^* = U$ is used. Finally, we get

$$\begin{aligned}
|\mathscr{M}(e^- e^+ \to \nu_m \bar{\nu}_m)|^2 = |\mathscr{M}_1|^2 &- 2\sum_i \sum_j U_{ei} U_{ej} U_{mi} U_{mj} \cos\left(\frac{\Delta m_{ij}^2 L}{2E}\right)\mathrm{Re}(\mathscr{M}_1^*\mathscr{M}_2) \\
&+ \left[\sum_i \sum_j U_{ei} U_{ej} U_{mi} U_{mj} \cos\left(\frac{\Delta m_{ij}^2 L}{2E}\right)\right]^2 |\mathscr{M}_2|^2
\end{aligned}$$