ANSWERS TO THE PROBLEMS IN

A First Course in String Theory

Solved by ZAN PAN

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ANSWERS TO THE PROBLEMS IN
A First Course in String Theory
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Part I

BASICS

Chapter 1

A Brief Introduction

■ Summary and Supplement

1. Two significant unifications
   - Maxwell's equations : unification of electricity and magnetism
   - Glashow-Weinberg-Salam model : unification of electromagnetism and the weak force
2. Two successful quantum theories
   - QED: the quantization of electromagnetism
   - QCD: the quantization of strong color force
3. The Standard Model
   - Quarks: (u, d), (c, s) and (t, b)
   - Leptons: (ν_e, e^-), (ν_μ, μ^-) and (ν_τ, τ^-)
   - Force carriers: $W^\pm$, Z^0, eight gluons and photon
4. Quantum gravity
   - String theory
   - Loop quantum gravity
   - Causal dynamical triangulation
   - Canonical general relativity
   - Noncommutative geometry
   - Twistor theory
5. String Theory
   - Each particle is identified as a particular vibrational mode of an elementary string.
   - String theory does not have adjustable dimensionless parameters.
   - The dimensionality of spacetime is fixed.
   - There are two kinds of strings: open strings and closed strings.
   - The graviton appears as a vibrational mode of closed strings.
   - Bosonic strings live in 26 dimensions while superstrings in 10.
   - M-theory is eleven-dimensional.
   - Our world is part of the D-branes.
   - The AdS/CFT correspondence is a remarkable physical equivalence between a certain four-dimensional gauge theory and a closed superstring theory.
   - String theory has made good strides towards a statistical mechanics interpretation of black hole entropy.

Chapter 2

Special Relativity and Extra Dimensions

■ Summary and Supplement

1. Intervals and Lorentz transformations

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{2.1}$$

$$a \cdot b = a^\mu b_\mu = \eta_{\mu\nu} a^\mu b^\nu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \tag{2.2}$$

$$x'^\mu = L^\mu x^\nu, \quad L^\mu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

2. Light-cone coordinates

$$x^+ = \frac{1}{2}(x^0 + x^1), \quad x^- = \frac{1}{2}(x^0 - x^1) \tag{2.4}$$

$$-ds^2 = \hat{\eta}_{\mu\nu} dx^\mu dx^\nu, \quad \hat{\eta}_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.5}$$

$$a \cdot b = a_+ b^+ + a_- b^- + a_2 b^2 + a_3 b^3, \quad a_+ = -a^-, \, a_- = -a^+ \tag{2.6}$$

$$p^+ = \frac{1}{2}(p^0 + p^1) = -p_-, \quad p^- = \frac{1}{2}(p^0 - p^1) = -p_+ \tag{2.7}$$

■ Quick Calculations

2.1 With the relation $\gamma = (1 - \beta^2)^{-1/2}$, it is easy to verify that

$$\begin{aligned}(x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 &= \gamma^2 \left[(x^0 - \beta x^1)^2 - (-\beta x^0 + x^1)^2 \right] - (x^2)^2 - (x^3)^2 \\ &= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2\end{aligned} \tag{2.8}$$

2.2 Using Eq. (2.2), we can obtain

$$\begin{aligned} a'^\mu b'^\mu &= -\gamma^2 (a^0 - \beta a^1)(b^0 - \beta b^1) + \gamma^2 (-\beta a^0 + a^1)(-\beta b^0 + b^1) + a^2 b^2 + a^3 b^3 \\ &= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \\ &= a^\mu b^\mu \end{aligned} \tag{2.9}$$

2.3 Suppose that $x^+ = \frac{1}{\sqrt{2}}(x^0 + x^1) = \gamma(x^0 - \beta x^1)$, then we have $\gamma = \frac{1}{\sqrt{2}}$ and $\beta = -1$. Therefore, such a transformation does not exist.

2.4 $\frac{E^2}{c^2} - \vec{p} \cdot \vec{p} = \gamma^2 m^2 c^2 - \gamma^2 m^2 v^2 = \gamma^2 (1 - \beta^2) m^2 c^2 = m^2 c^2$.

2.5 Consider the plain (x, y) with the identification $(x, y) \sim (x + 2\pi R, y + 2\pi R)$. The resulting space is a two-dimensional torus.

■ Solutions to the Problems

2.1 (a) $1\,\mathrm{C} = (10^9 \times 8.99 \times 10^9)^{\frac{1}{2}}\,\mathrm{esu} = 3 \times 10^9\,\mathrm{esu}$.

(b) If we set the Bolzmann's contant $k_B = 1$, then $[\mathrm{K}] = [\mathrm{N}\cdot\mathrm{m}] = ML^2T^{-2}$.

(c) $[e] = [\mathrm{esu}] = M^{1/2}L^{3/2}T^{-1}$, $[\hbar] = ML^2T^{-1}$, $[c] = LT^{-1}$, so we have

$$\frac{e^2}{\hbar c} = \frac{(1.602 \times 10^{-19} \times 3 \times 10^9)^2 \times 10^{-9}}{1.054 \times 10^{-34} \times 3 \times 10^8} \simeq \frac{1}{137}$$

In the Heaviside-Lorentz system of units, such a dimensionless number becomes $e^2/4\pi\hbar c$.

2.2 (a) According to Eq. (2.4), we have

$$x'^{+} = \frac{1}{\sqrt{2}}(x'^0 + x'^1) = \frac{1}{\sqrt{2}}\gamma(1-\beta)(x^0 + x^1) = \sqrt{\frac{1-\beta}{1+\beta}}x^+ \tag{2.10a}$$
$$x'^{-} = \frac{1}{\sqrt{2}}(x'^0 - x'^1) = \frac{1}{\sqrt{2}}\gamma(1+\beta)(x^0 - x^1) = \sqrt{\frac{1+\beta}{1-\beta}}x^- \tag{2.10b}$$
$$x'^2 = x^2 \tag{2.10c}$$
$$x'^3 = x^3 \tag{2.10d}$$

(b) From Eq. (2.4), we can obtain $x^0 = \frac{1}{\sqrt{2}}(x^+ + x^-)$, $x^1 = \frac{1}{\sqrt{2}}(x^+ - x^-)$. Then,

$$x'^{+} = \frac{1}{\sqrt{2}}(x^0 + \cos\theta\, x^1 - \sin\theta\, x^2) = \frac{1+\cos\theta}{2}x^+ + \frac{1-\cos\theta}{2}x^- - \frac{\sin\theta}{\sqrt{2}}x^2 \tag{2.11a}$$
$$x'^{-} = \frac{1}{\sqrt{2}}(x^0 - \cos\theta\, x^1 + \sin\theta\, x^2) = \frac{1-\cos\theta}{2}x^+ + \frac{1+\cos\theta}{2}x^- + \frac{\sin\theta}{\sqrt{2}}x^2 \tag{2.11b}$$
$$x'^2 = \sin\theta\, x^1 + \cos\theta\, x^2 = \frac{\sin\theta}{\sqrt{2}}(x^+ + x^-) + \cos\theta\, x^2 \tag{2.11c}$$
$$x'^3 = x^3 \tag{2.11d}$$

(c) This case differs strikingly from that of (a).

$$x'^{+} = \frac{1}{\sqrt{2}}\left[\gamma(x^0 - \beta x^3) + x^1\right] = \frac{\gamma+1}{2}x^+ + \frac{\gamma-1}{2}x^- - \frac{\gamma\beta}{\sqrt{2}}x^3 \tag{2.12a}$$
$$x'^{-} = \frac{1}{\sqrt{2}}\left[\gamma(x^0 - \beta x^3) - x^1\right] = \frac{\gamma-1}{2}x^+ + \frac{\gamma+1}{2}x^- - \frac{\gamma\beta}{\sqrt{2}}x^3 \tag{2.12b}$$
$$x'^2 = x^2 \tag{2.12c}$$
$$x'^3 = \gamma(-\beta x^0 + x^3) = -\frac{\gamma\beta}{\sqrt{2}}(x^+ + x^-) + \gamma x^3 \tag{2.12d}$$

2.3 (a) With the relations $a_0 = -a^0$, $a_1 = a^1$, $a_2 = a^2$ and $a_3 = a^3$, we can obtain

$$a'_0 = \gamma(a_0 + \beta a_1), \quad a'_1 = \gamma(\beta a_0 + a_1), \quad a'_2 = a_2, \quad a'_3 = a_3 \tag{2.13}$$

(b) We refer to the inverse Lorentz transformation.

$$\frac{\partial}{\partial x'^0} = \frac{\partial}{\partial x^0}\frac{\partial x^0}{\partial x'^0} + \frac{\partial}{\partial x^1}\frac{\partial x^1}{\partial x'^0} = \gamma(\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial x^1}) \tag{2.14a}$$
$$\frac{\partial}{\partial x'^1} = \frac{\partial}{\partial x^0}\frac{\partial x^0}{\partial x'^1} + \frac{\partial}{\partial x^1}\frac{\partial x^1}{\partial x'^1} = \gamma(\beta\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}) \tag{2.14b}$$
$$\frac{\partial}{\partial x'^2} = \frac{\partial}{\partial x^2} \tag{2.14c}$$
$$\frac{\partial}{\partial x'^3} = \frac{\partial}{\partial x^3} \tag{2.14d}$$

(c) Using the first quantization method, we have

$$p_\mu = (-\frac{E}{c}, p_x, p_y, p_z) = (-\frac{1}{c}i\hbar\frac{\partial}{\partial t}, -i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, -i\hbar\frac{\partial}{\partial z}) = \frac{\hbar}{i}\frac{\partial}{\partial x^\mu} \tag{2.15}$$

2.4 (a) The identification yields a semi-circle. There are two fixed points: $x = 0, 1$. A foundamental domain can be chosen as $[0, 1]$.

(b) As we know, $x = \pm 1$ are identified and $y = \pm 1$ are identified. Then it is obvious that there are four fixed points: $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$.

2.5 Let $\tan\theta = \beta$, then we can obtain the following from the inverse Lorentz transformation:

$$x^0 = \tan\theta\, x^1 + \gamma^{-1} x'^0, \quad x^0 = \cot\theta(x^1 - \gamma^{-1} x'^1). \tag{2.16}$$

It is obvious that the x'^0 and x'^1 axes appear in the original spacetime diagram as oblique axes. The angle between the x'^0 axis and the x^0 aixs is equal to that between the x'^1 axis and the x^1 axis, i.e. $\theta = \arctan\beta$. Diagrams for the axes are drawn in Fig. 2.1.

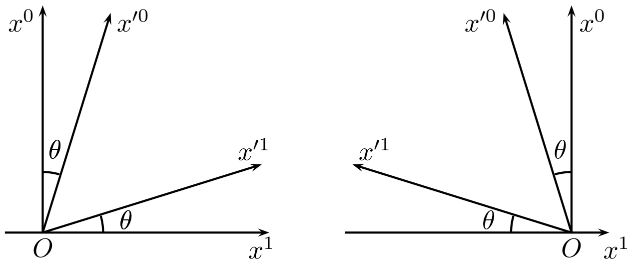

Fig. 2.1 The left illustrates how the axes appear when $\beta > 0$, while the right is for $\beta < 0$.

2.6 (a) In light-cone coordinates, it can be written as $(x^+, x^-) \sim (x^+, x^- - 2\sqrt{2}\pi R)$.

(b) With the relations $ct = \gamma(ct' + \beta x')$ and $x = \gamma(\beta ct' + x')$, we can obtain

$$\begin{pmatrix} \beta ct' + x' \\ ct' + \beta x' \end{pmatrix} \sim \begin{pmatrix} \beta ct' + x' \\ ct' + \beta x' \end{pmatrix} + \frac{2\pi}{\gamma}\begin{pmatrix} R \\ -R \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ ct' \end{pmatrix} \sim \begin{pmatrix} x' \\ ct' \end{pmatrix} + 2\pi\sqrt{\frac{1+\beta}{1-\beta}}\begin{pmatrix} R \\ -R \end{pmatrix} \tag{2.17}$$

(c) From the following, we can see that the velocity parameter $\beta = -R/\sqrt{R^2 + R_s^2}$ and that the compactification radius $R_c = R_s$.

$$\begin{pmatrix} \beta ct' + x' \\ ct' + \beta x' \end{pmatrix} \sim \begin{pmatrix} \beta ct' + x' \\ ct' + \beta x' \end{pmatrix} + \frac{2\pi}{\gamma}\begin{pmatrix} \sqrt{R^2 + R_s^2} \\ -R \end{pmatrix} \Rightarrow \begin{pmatrix} x' \\ ct' \end{pmatrix} \sim \begin{pmatrix} x' \\ ct' \end{pmatrix} + 2\pi\gamma\begin{pmatrix} \sqrt{R^2 + R_s^2} + \beta R \\ -R - \beta\sqrt{R^2 + R_s^2} \end{pmatrix}$$

(d) For example, $(0, 0)$ and $(-2\pi R, 2\pi\sqrt{R^2 + R_s^2})$ are related by the identification.

(e) Lightlike compactification with Radius R arises by boosting a standard compactification with radius R_s with Lorentz factor $\gamma \sim \sqrt{R^2 + R_s^2}/R_s$, in the limits as $R_s \to 0$.

2.7 (a) Using the result in 2.2 (a), we can rewrite the identification $(x^0, x^1) \sim (x'^0, x'^1)$ as

$$(x^+, x^-) \sim (e^{-\lambda} x^+, e^{\lambda} x^-), \quad \text{where } e^{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} \tag{2.18}$$

The range of λ is $(-\infty, \infty)$ and the orbifold fixed point is $(0, 0)$.

(b) The spacetime diagram refers to Fig. 2.2. From Eq. (2.18), we see that $e^{-\lambda} x^+ \cdot e^{\lambda} x^- = x^+ x^-$. So the idenfitication above relates points on the curves of $x^+ x^- = a^2$.

(c) $-ds^2 = -2dx^+ dx^- = -2dx^+ d(\frac{a^2}{x^+}) = 2(\frac{a}{x^+})^2 (dx^+)^2 > 0$. Therefore, the interval is spacelike.

(d) We choose $(e^{\lambda}, e^{2\lambda})$ as the integrating interval, then we have

$$\int_{e^{\lambda}}^{e^{2\lambda}} \frac{\sqrt{2}a}{x^-} dx^- = \sqrt{2}a\lambda \tag{2.19}$$

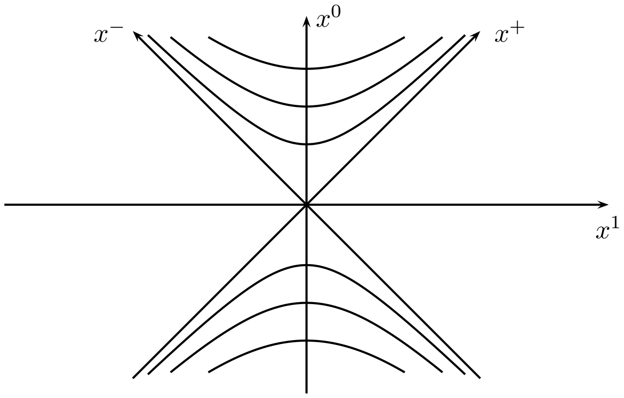

Fig. 2.2 The spacetime diagram for the $x^\pm$ axes and the family of curves $x^+x^- = a^2$.

2.8 (a) The energy eigenvalues are $E_{kl} = \frac{\hbar^2}{2m}\left[(\frac{k\pi}{a})^2 + (\frac{l}{R})^2\right]$, so we have

$$Z(a,R) = \int_0^\infty \int_0^\infty \exp\Big(-\frac{E_{kl}}{kT}\Big) dk\, dl = \frac{mkTaR}{2\hbar^2} \tag{2.20}$$

The results for a particle in a two-dimensional box with sides a and $2\pi R$ are the same.

(b) Since $R \ll a$, the lowest new energy level can be seen as E_{01}. Then, we have

$$E_{n0} < kT < E_{01} \Rightarrow \Big(\frac{n\pi}{a}\Big)^2 < \frac{2mkT}{\hbar^2} < \frac{1}{R^2} \tag{2.21}$$

And $Z(a,R)$ in this regime with the leading correction due to the small extra dimension is

$$Z(a,R) = \exp\Big(-\frac{\hbar^2}{2mkTR^2}\Big) \int_0^\infty \exp\Big[-\frac{\hbar^2}{2mkT}\Big(\frac{k\pi}{a}\Big)^2\Big] dk = \frac{a\sqrt{2mkT}}{\hbar} \exp\Big(-\frac{\hbar^2}{2mkTR^2}\Big)$$

Chapter 3

Electromagnetism and Gravitation in Various Dimensions

■ Summary and Supplement

1. Classical electrodynamics

   - Maxwell's equations in the Heaviside-Lorentz system of units

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \tag{3.1a}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{3.1b}$$

$$\nabla \cdot \boldsymbol{E} = \rho \tag{3.1c}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \tag{3.1d}$$

   - The Lorentz force

$$\frac{d\boldsymbol{p}}{dt} = q\big(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\big) \tag{3.2}$$

   - Vector and scalar potentials

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{3.3a}$$

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\Phi \tag{3.3b}$$

2. Relativistic electrodynamics

$$A^\mu = (\Phi, A^1, A^2, A^3), \quad A_\mu = (-\Phi, A^1, A^2, A^3) \tag{3.4}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{3.5}$$

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{3.6}$$

$$j^\mu = (c\rho, j^1, j^2, j^3), \quad \partial_\nu F^{\mu\nu} = \frac{1}{c}j^\mu \tag{3.7}$$

3. Gravitation and Planck's length

$$-\,ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \quad g^{\mu\alpha}(x)g_{\alpha\nu}(x) = \delta^\mu{}_\nu \tag{3.8}$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \quad \partial^2 h^{\mu\nu} - \partial_\alpha(\partial^\mu h^{\nu\alpha} + \partial^\nu h^{\mu\alpha}) + \partial^\mu\partial^\nu h = 0 \tag{3.9}$$

$$x^{\mu'} = x^\mu + \epsilon^\mu(x), \quad \delta h^{\mu\nu} = \partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu \tag{3.10}$$

$$\ell_{\rm P} = \sqrt{\frac{\hbar G}{c^3}} = 1.61 \times 10^{-33}\,\text{cm},\ t_{\rm P} = \sqrt{\frac{\hbar G}{c^5}} = 5.4 \times 10^{-44}\,\text{s},\ m_{\rm P} = \sqrt{\frac{\hbar c}{G}} = 2.17 \times 10^{-5}\,\text{g} \tag{3.11}$$

■ Quick Calculations

3.1 Under the gauge transformations, $\boldsymbol{E}$ is invariant:

$$\boldsymbol{E}' = -\frac{1}{c}\frac{\partial}{\partial t}\big(\boldsymbol{A} + \nabla\epsilon\big) - \nabla\big(\Phi - \frac{1}{c}\frac{\partial \epsilon}{\partial t}\big) = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\Phi = \boldsymbol{E} \tag{3.12}$$

3.2 As we know, $F_{\mu\nu}$ is antisymmetrical: $F_{\mu\nu} = -F_{\nu\mu}$. Therefore, we have

$$T_{\lambda\mu\nu} + T_{\mu\lambda\nu} = \partial_\lambda(F_{\mu\nu} + F_{\nu\mu}) + \partial_\mu(F_{\nu\lambda} + F_{\lambda\mu}) + \partial_\nu(F_{\lambda\mu} + F_{\mu\lambda}) = 0 \tag{3.13a}$$
$$T_{\lambda\mu\nu} + T_{\lambda\nu\mu} = \partial_\lambda(F_{\mu\nu} + F_{\nu\mu}) + \partial_\mu(F_{\nu\lambda} + F_{\lambda\mu}) + \partial_\nu(F_{\lambda\mu} + F_{\mu\lambda}) = 0 \tag{3.13b}$$

3.3 With the definition $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$, we can obtain

$$F^{\mu\nu} + F^{\nu\mu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} + \eta^{\nu\alpha}\eta^{\mu\beta}F_{\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta}(F_{\alpha\beta} + F_{\beta\alpha}) = 0 \tag{3.14a}$$
$$F^{0i} + F_{0i} = \eta^{0j}\eta^{ik}F_{jk} + F_{0i} = -\delta^{ik}F_{0k} + F_{0i} = 0 \tag{3.14b}$$
$$F^{ij} - F_{ij} = \eta^{ik}\eta^{jl}F_{kl} - F_{ij} = \delta^{ik}\delta^{jl}F_{kl} - F_{ij} = 0 \tag{3.14c}$$

The last one holds because the indices $i, j = 1, 2, 3$.

3.4 The following proof will use the property of the gamma function: $\Gamma(x + 1) = x\Gamma(x)$.

$$\text{vol}(S^{d-1}(R)) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}R^{d-1} = \frac{d}{dR}B^d(R) \Rightarrow B^d(R) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\frac{R^d}{d} = \frac{\pi^{d/2}}{\Gamma(1 + \frac{d}{2})}R^d \tag{3.15}$$

3.5 Since $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, it is easy to get $E(r) = \frac{q}{4\pi r^2}$.

3.6 $[F] = [q]^2 \cdot L^{d-1} \Rightarrow [q] = M^{1/2}L^{d/2}T^{-1}$.

3.7 Since $[G] = M^{-1}L^3T^{-2}$, $[c] = LT^{-1}$ and $[\hbar] = ML^2T^{-1}$, we have $-\alpha + \gamma = 0$, $3\alpha + \beta + 2\gamma = 1$, $-2\alpha - \beta - \gamma = 0$. Then, $\alpha = \gamma = 1/2$ and $\beta = -3/2$.

3.8 $E_\text{P} = m_\text{P}c^2 = 0.511 \times 10^{-3} \times 2.17 \times 10^{-5}/0.911 \times 10^{-27} = 1.22 \times 10^{19}\,\text{GeV}$.

3.9 $\oint \boldsymbol{g} \cdot d\boldsymbol{l} = \iint (\nabla \times \boldsymbol{g}) \cdot d\boldsymbol{S} = -\iint \nabla \times (\nabla V_g) \cdot d\boldsymbol{S} = 0$.

3.10 Since the units of $G^{(D)}\rho_m$ are the same in all dimensions, we have

$$[G^{(D)}]\frac{M}{L^{D-1}} = [G]\frac{M}{L^3} = \frac{[c]^3}{[\hbar]}\frac{M}{L} \Rightarrow \frac{G^{(D)}}{\big(\ell_\text{P}^{(D)}\big)^{D-1}} = \frac{c^3}{\hbar\ell_\text{P}^{(D)}} \Rightarrow \big(\ell_\text{P}^{(D)}\big)^{D-2} = \frac{\hbar G^{(D)}}{c^3} = (\ell_\text{P})^2\frac{G^{(D)}}{G}$$

■ Solutions to the Problems

3.1 We will check the case for $\mu = 1$. As we know, $\gamma ds = c\,dt$, $E_{10} = E_x$, $E_{11} = 0$, $E_{12} = B_z$, and $F_{13} = -B_y$, it is easy to obtain

$$\begin{aligned}\frac{dp_x}{dt} &= \frac{q}{c}\Big(F_{10}\frac{dx^0}{dt} + F_{11}\frac{dx^1}{dt} + F_{12}\frac{dx^2}{dt} + F_{13}\frac{dx^3}{dt}\Big)\\ &= \frac{q}{c}\Big(cE_x + B_z\frac{dy}{dt} - B_y\frac{dz}{dt}\Big) = q\big[E_x + \frac{1}{c}(v_yB_z - v_zB_y)\big]\end{aligned} \tag{3.16}$$

For $\mu = 0$, we have $F_{00} = 0$, $F_{01} = -E_x$, $F_{02} = -E_y$, $F_{03} = -E_z$ and $p_0 = -\frac{E}{c}$. Then,

$$-\frac{1}{c}\frac{dE}{dt} = -\frac{q}{c}(E_xv_x + E_yv_y + E_zv_z) \Rightarrow \frac{dE}{dt} = q\boldsymbol{E}\cdot\boldsymbol{v} = \boldsymbol{F}\cdot\boldsymbol{v} \tag{3.17}$$

Since $F_{\mu\nu}$ is invariant under the gauge transformation and p_μ and x^ν is independent of A_μ, it is a gauge invariant equation.

3.2 (a) T is nonvanishing only when each of its three indices takes a different value.

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = \frac{1}{c}\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \tag{3.18a}$$

$$\partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} = -\frac{1}{c}\frac{\partial B_y}{\partial t} + \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = 0 \tag{3.18b}$$

$$\partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} = \frac{1}{c}\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0 \tag{3.18c}$$

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{3.18d}$$

The first three are the components of Eq. (3.1a) and the last one is just Eq. (3.1b).

(b)

$$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{1}{c} j^0 = \rho \tag{3.19a}$$

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = -\frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \frac{1}{c} j^1 \tag{3.19b}$$

$$\frac{\partial F^{2\nu}}{\partial x^\nu} = -\frac{1}{c}\frac{\partial E_y}{\partial t} - \frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} = \frac{1}{c} j^2 \tag{3.19c}$$

$$\frac{\partial F^{3\nu}}{\partial x^\nu} = -\frac{1}{c}\frac{\partial E_z}{\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \frac{1}{c} j^3 \tag{3.19d}$$

The first one is just Eq. (3.1c) and the last three are the components of Eq. (3.1d).

3.3 (a) Using the ansata $E_z = B_x = B_y = 0$, we can easily obtain the following from the Maxwell's equations and the force law in four dimensions.

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{1}{c}\frac{\partial B_z}{\partial t}, \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \rho \tag{3.20}$$

$$\frac{\partial B_z}{\partial y} = \frac{1}{c} j^1 + \frac{1}{c}\frac{\partial E_x}{\partial t}, \quad -\frac{\partial B_z}{\partial x} = \frac{1}{c} j^2 + \frac{1}{c}\frac{\partial E_y}{\partial t} \tag{3.21}$$

(b) With the Lorentz covariant formulation, we have $A^\mu = (\Phi, A^1, A^2)$, $j^\mu = (c\rho, j^1, j^2)$, and

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & B_z \\ E_y & -B_z & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B_z \\ -E_y & -B_z & 0 \end{pmatrix} \tag{3.22}$$

Then $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ and $\partial_\nu F^{\mu\nu} = \frac{1}{c} j^\mu$ will give the same equations with thoes obtained in (a). And the relativistic form of the force law yields

$$\frac{dE}{dt} = q(E_x v_x + E_y v_y), \quad \frac{dp_x}{dt} = q(E_x + \frac{1}{c} v_y B_z), \quad \frac{dp_y}{dt} = q(E_y - \frac{1}{c} v_x B_z) \tag{3.23}$$

3.4 (a) Since A_μ is time-independent, we have $\partial_0 F_{ij} = 0$. Then,

$$T_{0ij} = \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = \partial_i E_j - \partial_j E_i = 0 \tag{3.24}$$

This condition is satisfied because $\nabla \times \boldsymbol{E} = -\nabla \times (\nabla \Phi) = 0$.

(b) With the relations $\mathrm{vol}(S^{d_1}(r)) = 2\pi^{d/2} r^{d-1}/\Gamma(\frac{d}{2})$ and $\boldsymbol{E} = -d\Phi/dr$, we have

$$\iint \boldsymbol{E} \cdot d\boldsymbol{S} = q \Rightarrow E(r) = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}} \frac{q}{r^{d-1}} \Rightarrow \Phi(r) = \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}} \frac{q}{r^{d-2}} \tag{3.25}$$

3.5 $\nabla \cdot \boldsymbol{f} = \nabla f \cdot \hat{\boldsymbol{r}} + f \nabla \cdot (\frac{\boldsymbol{r}}{r}) = f'(r) + f(r)[\nabla(\frac{1}{r}) \cdot \boldsymbol{r} + \frac{1}{r} \nabla \cdot \boldsymbol{r}] = f'(r) + \frac{d-1}{r} f(r)$.

3.6 By the definition, we have

$$\begin{aligned} &\int_0^1 dt\, t^{z-1}\Big(e^{-t} - \sum_{n=0}^{N} \frac{n!}{(-1)^n}\Big) + \sum_{n=0}^{N} \frac{n!}{(-1)^n} \frac{1}{z+n} + \int_1^\infty dt\, e^{-t} t^{z-1} \\ &= \Gamma(z) + \sum_{n=0}^{\infty} \frac{n!}{(-1)^n}\Big(\frac{1}{z+n} - \int_0^1 t^{n+z-1} dt\Big) = \Gamma(z) \end{aligned} \tag{3.26}$$

The function of the first integral has the order of $\mathcal{O}(t^{z+N})$. For $\Re(z) > -N-1$, the integral on $[0,1]$ will always converge. So the right-hand side above is well defined. And also, we can obtain the following identity:

$$\Gamma(z) = \frac{\Gamma(z+N+1)}{z(z+1)\cdots(z+N)} \tag{3.27}$$

Obviously, $z = 0, -1, -2, \ldots$, are poles for $\Gamma(z)$. The value of residue at $z = -n$ is

$$\mathrm{Res}[\Gamma(z), -n] = \lim_{z\to -n}(z+n)\Gamma(z) = \frac{(-1)^n}{n!} \tag{3.28}$$

3.7 (a) The "gravitational" Bohr radius for a hydrogen atom is $\hbar^2/Gm^3 = 2.2\times 10^{32}\,\mathrm{m}$.

(b) Suppose $kT = (8\pi M)^{-1}G^\alpha c^\beta \hbar^\gamma$, then we have

$$\begin{aligned}[E] = [G]^\alpha [c]^\beta [\hbar]^\gamma M^{-1} &\Rightarrow ML^2T^{-2} = M^{-\alpha+\gamma-1}L^{3\alpha+\beta+2\gamma}T^{-2\alpha-\beta-\gamma} \\ &\Rightarrow \alpha = -1, \beta = 3, \gamma = 1\end{aligned} \tag{3.29}$$

For $M = 10^6\,M_\odot$, $M_\odot = 2\times 10^{30}\,\mathrm{kg}$, the temperature is $T = 6.15\times 10^{-14}\,\mathrm{K}$. And for the black hole whose temperature is room temperature (300K), its mass will be $M = 4.2\times 10^{20}\,\mathrm{kg}$.

3.8 We use the effective potential $V_{\text{eff}}(r) = V_g(r) + J^2/2mr^2$ to discuss the planetary motion.

$$\begin{aligned}\int_{S^{d-1}} \boldsymbol{g}\cdot d\boldsymbol{S} = \int_{B^d}\nabla\cdot\boldsymbol{g}\,d(\mathrm{vol}) = -4\pi G^{(D)}m \Rightarrow g(r) &= -\frac{2\Gamma(\frac{d}{2})}{\pi^{d/2-1}}\frac{G^{(D)}m}{r^{d-1}} = -\frac{d}{dr}V_g(r) \\ \Rightarrow V_g(r) &= -\frac{\Gamma(\frac{d}{2}-1)}{\pi^{d/2-1}}\frac{G^{(D)}m}{r^{d-2}}\end{aligned} \tag{3.30}$$

From the condition $\frac{d}{dr}\big[V_{\text{eff}}(r)\big]\big|_{r=r_0} = 0$, we can solve r_0. Then,

$$\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_0} = \frac{(4-d)J^2}{mr_0^4} \tag{3.31}$$

For $d = 3$, it is positive, so the planetary circular orbits in the four-dimensional world are stable under perturbations; while for $d \geq 4$, they are not stable.

3.9 (a) Using the result in Eq. (3.30), we can directly write down the expression:

$$V_g^{(5)}(r) = -\frac{G^{(5)}M}{\pi r^2} \tag{3.32}$$

(b) The circel can be constructed by the identification of $\mathbb{R}^1$: $w \sim w + 2n\pi a$, thus we have

$$V_g^{(5)}(x,y,z,0) = -\sum_{n=-\infty}^{\infty}\frac{G^{(5)}M}{\pi[R^2+(2n\pi a)^2]} \tag{3.33}$$

(c) For $R \gg a$, the potential becomes

$$V_g^{(5)}(x,y,z,0) = -\int_{-\infty}^{\infty}\frac{G^{(5)}M}{\pi}\frac{dt}{R^2+(2\pi a t)^2} = -\frac{G^{(5)}M}{2\pi a R} = -\frac{GM}{R}, \tag{3.34}$$

in which we have used the relation $G^{(5)} = 2\pi a G$.

3.10 (a) Using the following identity

$$\sum_{n=-\infty}^{\infty}\frac{1}{1+(n\pi x)^2} = \frac{1}{x}\coth\left(\frac{1}{x}\right), \tag{3.35}$$

we can find an exact closed-form expression for the potential:

$$V_g^{(5)}(x,y,z,0) = -\frac{G^{(5)}M}{\pi R^2}\frac{R}{2a}\coth\left(\frac{R}{2a}\right) = -\frac{GM}{R}\coth\left(\frac{R}{2a}\right) \tag{3.36}$$

(b) For $R \gg a$, we can expand the result above to the leading correction:

$$V_g^{(5)}(x, y, z, 0) = -\frac{GM}{R}\frac{1+e^{-\lambda}}{1-e^{-\lambda}} \simeq -\frac{GM}{R}\big(1 + 2e^{-\lambda}\big), \tag{3.37}$$

where $\lambda = R/a$. When $\lambda = 5.3$, the correction is of order 1%.

(c) Using the following identity

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + \cdots, \quad 0 < |x| < \pi, \tag{3.38}$$

we can expand the gravitational potential when $R \ll a$:

$$V_g^{(5)}(x, y, z, 0) \simeq -\frac{GM}{R}\Big(\frac{2a}{R} + \frac{R}{6a}\Big) = -\frac{G^{(5)}M}{\pi R^2}\Big(1 + \frac{R^2}{12a^2}\Big) \tag{3.39}$$

The first term has the same form to the gravitational potential discussed in 3.9 (a) .

Chapter 4

Nonrelativistic Strings

■ Summary and Supplement

1. Equations of motion for transverse oscillations

$$\frac{\partial^2 y}{\partial t^2} = v_0^2 \frac{\partial^2 y}{\partial x^2}, \quad v_0 = \sqrt{\frac{T_0}{\mu_0}} \tag{4.1}$$

$$y(t, x) = h_+(x - v_0 t) + h_-(x + v_0 t) \tag{4.2}$$

2. The nonrelativistic string Lagrangian

$$\mathcal{L}\Big(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\Big) = \frac{1}{2}\mu_0\Big(\frac{\partial y}{\partial t}\Big)^2 - \frac{1}{2}T_0\Big(\frac{\partial y}{\partial x}\Big)^2 \tag{4.3}$$

$$\mathcal{P}^t = \frac{\partial \mathcal{L}}{\partial \dot{y}},\ \mathcal{P}^x = \frac{\partial \mathcal{L}}{\partial y'}, \quad \frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x} = 0 \tag{4.4}$$

■ Quick Calculations

4.1 As we vary: $y(t, x) \to y(t, x) + \delta y(t, x)$, the variation is

$$\delta S = \delta \int_{t_i}^{t_f} dt \int_0^a dx \Big[\frac{1}{2}\mu_0\Big(\frac{\partial y}{\partial t}\Big)^2 - \frac{1}{2}T_0\Big(\frac{\partial y}{\partial x}\Big)^2\Big] = \int_{t_i}^{t_f} dt \int_0^a dx \Big[\mu_0 \frac{\partial y}{\partial t}\frac{\partial(\delta y)}{\partial t} - T_0 \frac{\partial y}{\partial x}\frac{\partial(\delta y)}{\partial x}\Big]$$

4.2 When we vary the motion by δy, the variation of the action is given by

$$\begin{aligned}
\delta S &= \int_{t_i}^{t_f} dt \int_0^a dx \big(\mathcal{P}^t \delta\dot{y} + \mathcal{P}^x \delta y'\big) \\
&= \int_{t_i}^{t_f} dt \int_0^a dx \Big[\frac{\partial}{\partial t}(\mathcal{P}^t \delta y) - \frac{\partial \mathcal{P}^t}{\partial t}\delta y + \frac{\partial}{\partial x}(\mathcal{P}^x \delta y) - \frac{\partial \mathcal{P}^x}{\partial x}\delta y\Big] \\
&= \int_0^a \Big(\mathcal{P}^t \delta y\Big)\Big|_{t=t_i}^{t=t_f} dx + \int_{t_i}^{t_f} \Big(\mathcal{P}^x \delta y\Big)\Big|_{x=0}^{x=a} dt - \int_{t_i}^{t_f} dt \int_0^a dx \Big(\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x}\Big)\delta y
\end{aligned} \tag{4.5}$$

4.3 With the relations $\mathcal{P}^t = \mu_0 \dot{y}$ and $\mathcal{P}^x = -T_0 y'$, we can rewrite Eq. (4.5) as

$$\begin{aligned}
\delta S &= \int_0^a \Big(\mathcal{P}^t \delta y\Big)\Big|_{t=t_i}^{t=t_f} dx + \int_{t_i}^{t_f} \Big(\mathcal{P}^x \delta y\Big)\Big|_{x=0}^{x=a} dt - \int_{t_i}^{t_f} dt \int_0^a dx \Big(\frac{\partial \mathcal{P}^t}{\partial t} + \frac{\partial \mathcal{P}^x}{\partial x}\Big)\delta y \\
&= \int_0^a \Big(\mu_0 \frac{\partial y}{\partial t}\delta y\Big)\Big|_{t=t_i}^{t=t_f} dx + \int_{t_i}^{t_f} \Big(-T_0 \frac{\partial y}{\partial x}\delta y\Big)\Big|_{x=0}^{x=a} dt - \int_{t_i}^{t_f} dt \int_0^a dx \Big(\mu_0 \frac{\partial^2 y}{\partial t^2} - T_0 \frac{\partial^2 y}{\partial x^2}\Big)\delta y
\end{aligned}$$

■ Solutions to the Problems

4.1 For small oscillations, the horizontal force dF_h is

$$dF_h = T_0\Big[1 + \Big(\frac{\partial y}{\partial x}\Big)^2\Big]_{x+dx}^{-\frac{1}{2}} - T_0\Big[1 + \Big(\frac{\partial y}{\partial x}\Big)^2\Big]_{x}^{-\frac{1}{2}} = -T_0 \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x^2} dx \tag{4.6}$$

Since $\partial y / \partial x \ll 1$, it is much smaller than the vertical force dF_v.

4.2 For the small longitudinal oscillations, the equation can be derived as

$$T(x+dx)-T(x)=\tau_0\frac{\partial z}{\partial x}\Big|_{x+dx}-\tau_0\frac{\partial z}{\partial x}\Big|_{x}=\tau_0\frac{\partial^2 z}{\partial x^2}dx=\mu_0\frac{\partial^2 z}{\partial t^2}\Rightarrow\frac{\partial^2 z}{\partial t^2}=\frac{\tau_0}{\mu_0}\frac{\partial^2 z}{\partial x^2}\tag{4.7}$$

Thus, the velocity of the waves is $\sqrt{\tau_0/\mu_0}$.

4.3 (a) Suppose $u=-v_0t$ and $w=u-a$, then we can obtain the following:

$$y(t,0)=h_+(-v_0t)+h_-(v_0t)=0\Rightarrow h_+(u)=-h_-(-u)\tag{4.8}$$

$$y(t,a)=h_+(a+u)+h_-(a-u)=0\Rightarrow h_+(u+a)-h_+(u-a)=0\Rightarrow h_+(w)=h_+(w+2a)\tag{4.9}$$

(b) From the initial conditions, we have

$$y|_{t=0}=h_+(x)+h_-(x)=0,\quad \frac{\partial y}{\partial t}\Big|_{t=0}=-v_0h'_+(x)+v_0h'_-(x)\tag{4.10}$$

The second equation can be rewritten by the integrating in two different intevals, which gives different functions of $h_+(u)$. For $0<x<a$, we have

$$-h_+(x)+h_-(x)=\int_0^x\frac{\xi}{a}\Big(1-\frac{\xi}{a}\Big)d\xi\Rightarrow h_+(u)=\frac{1}{2}\Big(\frac{u^3}{3a^2}-\frac{u^2}{2a}-c\Big)\tag{4.11}$$

For $-a<x<0$, it becomes

$$-h_+(x)+h_-(x)=\int_0^x-\frac{\xi}{a}\Big(1+\frac{\xi}{a}\Big)d\xi\Rightarrow h_+(u)=\frac{1}{2}\Big(\frac{u^3}{3a^2}+\frac{u^2}{2a}+c\Big)\tag{4.12}$$

We can extand $h_+(u)$ for all u with the periodic conditions $h_+(u)=h_+(u+2a)$.

(c) For x and v_0t in the domain $D=\{(x,v_0t)|\,0\le x\pm v_0t<a\}$, the wave function is

$$y(t,x)=\frac{1}{2}\Big[\frac{(x-v_0t)^3}{3a^2}-\frac{(x-v_0t)^2}{2a}\Big]-\frac{1}{2}\Big[\frac{(x+v_0t)^3}{3a^2}-\frac{(x+v_0t)^2}{2a}\Big]=v_0t\Big(\frac{x}{a}-\frac{x^2}{a^2}-\frac{v_0^2t^2}{3a^2}\Big)\tag{4.13}$$

(d) From the function above, we can obtain

$$\frac{\partial y}{\partial t}=v_0\Big(\frac{x}{a}-\frac{x^2}{a^2}-\frac{v_0^2t^2}{a^2}\Big)=-v_0\Big[\frac{1}{4}-\Big(\frac{x}{a}-\frac{1}{2}\Big)^2\Big]-\frac{v_0^2t^2}{a^2}\tag{4.14}$$

Obviously, at $t=0$ the midpoint $x=a/2$ has the largest velocity. It is easy to concluce that the velocity of the midpoint reaches zero at $t_0=a/2v_0$ and $y(t_0,a/2)=a/12$.

4.4

4.5

4.6 (a) The variation δS of the action under a variation δq of the coordinate can be derived as

$$\begin{aligned}\delta S=\delta\int dt\,L(q(t),\dot q(t);t)&=\int dt\Big(\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial\dot q}\delta\dot q\Big)\\&=\int dt\,\Big[\frac{\partial L}{\partial q}\delta q+\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot q}\delta q\Big)-\Big(\frac{d}{dt}\frac{\partial L}{\partial\dot q}\Big)\delta q\Big]\\&=\int dt\,\Big(\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial\dot q}\Big)\delta q\end{aligned}\tag{4.15}$$

Then, $\delta S=0$ gives the famous Euler-Lagrange equation

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial\dot q}=0\tag{4.16}$$

(b) The derivation is similar to that in part (a):

$$\begin{aligned}\delta S=\delta\int d^Dx\,\mathcal{L}(\phi(x),\partial_\mu\phi(x))&=\int d^Dx\Big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta(\partial_\mu\phi)\Big]\\&=\int d^Dx\Big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\Big)-\Big(\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Big)\delta\phi\Big]\\&=\int d^Dx\Big(\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\Big)\delta\phi\end{aligned}\tag{4.17}$$

Then we can obtain the Euler-Lagrange equation for the dynamical field $\phi(x)$

$$\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}=0\tag{4.18}$$

Chapter 5

The Relativistic Point Particle

■ Summary and Supplement

1. Action for a relativistic point particle

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}}, \quad H = \frac{mc^2}{\sqrt{1-v^2/c^2}}, \quad \boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = \frac{m\boldsymbol{v}}{\sqrt{1-v^2/c^2}} \tag{5.1}$$

$$x^\mu = x^\mu(\tau), \quad S = -mc\int ds = -mc\int_{\tau_i}^{\tau_f}\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\,d\tau \tag{5.2}$$

2. Equations of motion

$$\delta(dx^\mu) = d(\delta x^\mu), \quad \frac{dp_\mu}{d\tau} = 0, \quad \frac{d^2x^\mu}{ds^2} = 0 \tag{5.3}$$

$$S = -mc\int_{\mathcal{P}} ds + \frac{q}{c}\int_{\mathcal{P}} A_\mu(x)\,dx^\mu \tag{5.4}$$

■ Quick Calculations

5.1 Since x_i and x_f are fixed under the variation, we have

$$\delta S_{\mathrm{nr}} = \delta\int\frac{1}{2}mv^2\,dt = \int m\boldsymbol{v}\cdot\boldsymbol{v}_0\,dt = m\boldsymbol{v}_0\cdot\int d\boldsymbol{x} = 0 \tag{5.5}$$

5.2 For any arbitrary parameter $\tau'(\tau)$, we have

$$\frac{dp_\mu}{d\tau} = \frac{dp_\mu}{d\tau'}\frac{d\tau'}{d\tau} = 0 \Rightarrow \frac{dp_\mu}{d\tau'} = 0 \tag{5.6}$$

■ Solutions to the Problems

5.1 For the new parameter $\tau = f(s)$, we have

$$\frac{d^2x^\mu}{ds^2} = \frac{d}{ds}\Big(\frac{dx^\mu}{d\tau}\frac{df}{ds}\Big) = \frac{d^2x^\mu}{d\tau^2}\Big(\frac{df}{ds}\Big)^2 + \frac{dx^\mu}{d\tau}\frac{d^2f}{ds^2} = 0 \Rightarrow \frac{d^2f}{ds^2} = 0 \Rightarrow f = as+b, \tag{5.7}$$

where a and b are constants independent of s.

5.2 If we reexpress the integrand ds using the parameterized world-line, i.e.

$$ds = \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\,d\tau \tag{5.8}$$

then we can obtain the following

$$\frac{d^2x^\lambda}{ds^2} = \frac{d^2x^\lambda}{d\tau^2}\Big(\frac{d\tau}{ds}\Big)^2 + \frac{dx^\lambda}{d\tau}\frac{d^2\tau}{ds^2} = 0 \Rightarrow \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{d^2x^\lambda}{d\tau^2} - \frac{1}{2}\frac{dx^\lambda}{d\tau}\frac{d}{d\tau}\Big(\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Big) = 0 \tag{5.9}$$

5.3

5.4 (a) Using the relations $A_\mu = (-\Phi, \boldsymbol{A})$ and $dx^\mu/dt = (c, \boldsymbol{v})$, we can rewrite the action S as

$$S = \int \frac{1}{2}mv^2\,dt + \int \frac{q}{c}(-\Phi c + \boldsymbol{A}\cdot\boldsymbol{v})\,dt \Rightarrow L = \frac{1}{2}mv^2 - q\Phi + \frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{v} \tag{5.10}$$

(b) By the definition, it is easy to get

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + \frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{v} \tag{5.11}$$

(c) The Hamiltonian for the charged particle is given by

$$H = \boldsymbol{p}\cdot\boldsymbol{v} - L = mv^2 + \frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{v} - \Big(\frac{mv^2}{2} - q\Phi + \frac{q}{c}\boldsymbol{A}\cdot\boldsymbol{v}\Big) = \frac{mv^2}{2} + q\Phi = \frac{1}{2m}\Big(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\Big)^2 + q\Phi \tag{5.12}$$

5.5 The variation of the action for a charged point particle can be derived as

$$\begin{aligned}
\delta S &= -\int_{\tau_i}^{\tau_f} d\tau\,\delta x^\mu \frac{dp_\mu}{d\tau} + \frac{q}{c}\int_{\tau_i}^{\tau_f} d\tau\Big[\delta A_\mu \frac{dx^\mu}{d\tau} + A_\mu \delta\Big(\frac{dx^\mu}{d\tau}\Big)\Big] \\
&= -\int_{\tau_i}^{\tau_f} d\tau\,\delta x^\mu \frac{dp_\mu}{d\tau} + \frac{q}{c}\int_{\tau_i}^{\tau_f} d\tau\Big[\frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu \frac{dx^\mu}{d\tau} + \frac{d}{d\tau}(A_\mu \delta x^\mu) - \frac{dA^\mu}{d\tau}\delta x^\mu\Big] \\
&= -\int_{\tau_i}^{\tau_f} d\tau\,\delta x^\mu \frac{dp_\mu}{d\tau} + \frac{q}{c}\int_{\tau_i}^{\tau_f} d\tau\Big(\frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu \frac{dx^\mu}{d\tau} - \frac{\partial A^\mu}{\partial x^\nu}\frac{dA^\nu}{d\tau}\delta x^\mu\Big) \\
&= -\int_{\tau_i}^{\tau_f} d\tau\,\delta x^\mu \frac{dp_\mu}{d\tau} + \frac{q}{c}\int_{\tau_i}^{\tau_f} d\tau\Big(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu}\Big)\frac{dx^\nu}{d\tau}\delta x^\mu \\
&= \int_{\tau_i}^{\tau_f} d\tau\Big(-\frac{dp_\mu}{d\tau} + \frac{q}{c}F_{\mu\nu}\frac{dx^\nu}{d\tau}\Big)\delta x^\mu
\end{aligned} \tag{5.13}$$

Then, $\delta S = 0$ gives the equation of motion:

$$\frac{dp_\mu}{d\tau} = \frac{q}{c}F_{\mu\nu}\frac{dx^\nu}{d\tau} \tag{5.14}$$

5.6

5.7

Chapter 6

Relativistic Strings

■ Summary and Supplement

1. Area functional for spatial surfaces

$$A = \int d\xi^1 d\xi^2 \sqrt{\Big(\frac{\partial \boldsymbol{x}}{\partial \xi^1}\cdot\frac{\partial \boldsymbol{x}}{\partial \xi^1}\Big)\Big(\frac{\partial \boldsymbol{x}}{\partial \xi^2}\cdot\frac{\partial \boldsymbol{x}}{\partial \xi^2}\Big) - \Big(\frac{\partial \boldsymbol{x}}{\partial \xi^1}\cdot\frac{\partial \boldsymbol{x}}{\partial \xi^2}\Big)^2} \tag{6.1}$$

$$g_{ij} = \frac{\partial \boldsymbol{x}}{\partial \xi^i}\cdot\frac{\partial \boldsymbol{x}}{\partial \xi^i}, \quad ds^2 = g_{ij} d\xi^i d\xi^j, \quad A = \int d\xi^1 d\xi^2 \sqrt{g}, \quad g = \det(g_{ij}) \tag{6.2}$$

2. The Nambu-Goto action

$$A = \int d\tau d\sigma \sqrt{\Big(\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial \sigma}\Big)^2 - \Big(\frac{\partial X}{\partial \tau}\Big)^2\Big(\frac{\partial X}{\partial \sigma}\Big)^2} \tag{6.3}$$

$$S = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2} \tag{6.4}$$

$$\gamma_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X_\mu}{\partial \xi^\alpha}\frac{\partial X_\nu}{\partial \xi^\beta}, \quad S = -\frac{T_0}{c}\int d\tau d\sigma \sqrt{-\gamma}, \quad \gamma = \det(\gamma_{\alpha\beta}) \tag{6.5}$$

3. Equation of motion, boundary conditions, and D-branes

$$\mathcal{L}(\dot{X}^\mu, X^{\mu\prime}) = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2} \tag{6.6}$$

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}} \tag{6.7}$$

$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (X')^2 X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}} \tag{6.8}$$

$$\text{Equation for relativistic string: } \frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0 \tag{6.9}$$

$$\text{Dirichlet boundary condition: } \frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0, \quad \mu \neq 0, \ \sigma_* = 0 \text{ or } \sigma_1 \tag{6.10}$$

$$\text{Free endpoint condition: } \mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0, \quad \sigma_* = 0 \text{ or } \sigma_1 \tag{6.11}$$

4. Action in terms of transverse velocity

$$X^\mu(\tau,\sigma) = (ct, \boldsymbol{X}(t,\sigma)), \quad \frac{\partial X^\mu}{\partial \tau} = (c, \frac{\partial \boldsymbol{X}}{\partial t}), \quad \frac{\partial X^\mu}{\partial \sigma} = (0, \frac{\partial \boldsymbol{X}}{\partial \sigma}) \tag{6.12}$$

$$\boldsymbol{v}_\perp = \frac{\partial \boldsymbol{X}}{\partial t} - \Big(\frac{\partial \boldsymbol{X}}{\partial t}\cdot\frac{\partial \boldsymbol{X}}{\partial s}\Big)\frac{\partial \boldsymbol{X}}{\partial s}, \quad v_\perp^2 = \Big(\frac{\partial \boldsymbol{X}}{\partial t}\Big)^2 - \Big(\frac{\partial \boldsymbol{X}}{\partial t}\cdot\frac{\partial \boldsymbol{X}}{\partial s}\Big)^2 \tag{6.13}$$

$$S = -T_0 \int dt \int_0^{\sigma_1} d\sigma \Big(\frac{ds}{d\sigma}\Big)\sqrt{1 - \frac{v_\perp^2}{c^2}}, \quad L = -T_0 \int ds \sqrt{1 - \frac{v_\perp^2}{c^2}} \tag{6.14}$$

5. Motion of open string endpoints

$$\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c^2}\Big(1-\frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}}\Big\{\Big(\frac{\partial \boldsymbol{X}}{\partial s}\cdot\frac{\partial \boldsymbol{X}}{\partial t}\Big)\dot{X}^\mu + \Big[c^2-\Big(\frac{\partial \boldsymbol{X}}{\partial t}\Big)^2\Big]\frac{\partial X^\mu}{\partial s}\Big\} \tag{6.15}$$

$$\mathcal{P}^{\sigma 0} = -\frac{T_0}{c}\Big(1-\frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}}\Big(\frac{\partial \boldsymbol{X}}{\partial s}\cdot\frac{\partial \boldsymbol{X}}{\partial t}\Big) = 0 \Rightarrow \boldsymbol{v}\cdot\frac{\partial \boldsymbol{X}}{\partial s} = 0 \tag{6.16}$$

$$\mathcal{P}^{\sigma\mu} = -T_0\Big(1-\frac{v^2}{c^2}\Big)^{\frac{1}{2}}\frac{\partial X^\mu}{\partial s} = 0 \Rightarrow v^2 = c^2 \tag{6.17}$$

■ Quick Calculations

6.1 Using the chain rule of derivatives, we can obtain

$$\begin{aligned} A &= \int d\xi^1 d\xi^2 \sqrt{\Big(\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^1}\cdot\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^1}\Big)\Big(\frac{\partial\tilde{\xi}^1}{\partial\xi^1}\Big)^2\Big(\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^2}\cdot\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^2}\Big)\Big(\frac{\partial\tilde{\xi}^2}{\partial\xi^2}\Big)^2 - \Big(\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^1}\cdot\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^2}\Big)^2\Big(\frac{\partial\tilde{\xi}^1}{\partial\xi^1}\frac{\partial\tilde{\xi}^2}{\partial\xi^2}\Big)^2} \\ &= \int d\tilde{\xi}^1 d\tilde{\xi}^2 \sqrt{\Big(\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^1}\cdot\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^1}\Big)\Big(\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^2}\cdot\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^2}\Big) - \Big(\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^1}\cdot\frac{\partial \boldsymbol{x}}{\partial\tilde{\xi}^2}\Big)^2} \end{aligned} \tag{6.18}$$

6.2 Using the chain rule, we can prove that

$$M_{ij}\tilde{M}_{jk} = \frac{\partial\xi^i}{\partial\tilde{\xi}^j}\frac{\partial\tilde{\xi}^j}{\partial\xi^k} = \frac{\partial\xi^i}{\partial\xi^k} = \delta^i{}_k, \quad \tilde{M}_{ij}M_{jk} = \frac{\partial\tilde{\xi}^i}{\partial\xi^j}\frac{\partial\xi^j}{\partial\tilde{\xi}^k} = \frac{\partial\tilde{\xi}^i}{\partial\tilde{\xi}^k} = \delta^i{}_k \tag{6.19}$$

6.3 For a point on the world-sheet where all tangent vectors are spcelike with the exception of one that is null, we have

$$\frac{\partial X}{\partial\tau} = \frac{\partial X}{\partial\sigma} \Rightarrow v^2(\lambda) = (\lambda+1)^2\frac{\partial X}{\partial\sigma} \geq 0 \tag{6.20}$$

When $\lambda = -1$, the tangent vector $v = 0$.

6.4 With the relations $\dot{X}\cdot X' = \dot{X}^\mu X'_\mu$, $(\dot{X})^2 = \dot{X}^\mu\dot{X}_\mu$, and $(X')^2 = X^{\mu\prime}X'_\mu$, it is easy to verify Eq. (6.7) and Eq. (6.8).

■ Solutions to the Problems

6.1 Since the oscillations are small, we have

$$ds^2 = d\boldsymbol{X}\cdot d\boldsymbol{X} = (dx, d\boldsymbol{y})\cdot(dx, d\boldsymbol{y}) = dx^2 + d\boldsymbol{y}\cdot d\boldsymbol{y} \simeq dx^2 \tag{6.21}$$

Then the following approximation holds:

$$\boldsymbol{v}_\perp = \frac{\partial \boldsymbol{X}}{\partial t} - \Big(\frac{\partial \boldsymbol{X}}{\partial t}\cdot\frac{\partial \boldsymbol{X}}{\partial s}\Big)\frac{\partial \boldsymbol{X}}{\partial s} \simeq \Big(\frac{\partial x}{\partial t}, \frac{\partial \boldsymbol{y}}{\partial t}\Big) - \Big(\frac{\partial x}{\partial t}, \frac{\partial \boldsymbol{y}}{\partial t}\Big)\cdot(1,0)(1,0) = \Big(0, \frac{\partial \boldsymbol{y}}{\partial t}\Big) \tag{6.22}$$

Furthermore, the action reduces to be

$$S \simeq -T_0\int dt\int dx\sqrt{1-\frac{1}{c^2}\Big(\frac{\partial \boldsymbol{y}}{\partial t}\Big)^2} \simeq -T_0\int_{t_i}^{t_f} dt\int_0^a dx\Big[1-\frac{1}{2c^2}\Big(\frac{\partial \boldsymbol{y}}{\partial t}\Big)^2\Big] \tag{6.23}$$

Up to an additive constant $-aT_0(t_f-t_i)$, it is just the action for a nonrelativistic string performing small transverse oscillations, since $\mu_0 = T_0/c^2$.

6.2 We start our derivation with the Nambu-Goto action and work in the static gauge:

$$\begin{aligned} S &\simeq -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau\int_0^{\sigma_1} d\sigma\sqrt{0-\Big(\frac{\partial \boldsymbol{X}}{\partial\sigma}\Big)^2(-c^2)} \\ &= -T_0\int_{t_i}^{t_f} dt\int_0^1 d\sigma\Big|\frac{\partial \boldsymbol{X}}{\partial\sigma}\Big| \\ &= -T_0\int_{t_i}^{t_f} dt\int_0^a dx\sqrt{1+\Big(\frac{\partial \boldsymbol{y}}{\partial x}\Big)^2} \\ &\simeq -T_0\int_{t_i}^{t_f} dt\int_0^a dx\Big[1+\frac{1}{2}\Big(\frac{\partial \boldsymbol{y}}{\partial x}\Big)^2\Big] \\ &= -aT_0(t_f-t_i) - T_0\int_{t_i}^{t_f} dt\int_0^a dx\,\frac{1}{2}\Big(\frac{\partial \boldsymbol{y}}{\partial x}\Big)^2 \end{aligned} \tag{6.24}$$

6.3

6.4 In this problem, we are discussiong the time evolution of a closed circular string. It is clear to us that $\dot{R}(t) = v_\perp$, so we have

$$L = -T_0 \int ds \sqrt{1 - \dot{R}^2(t)/c^2} = -2\pi R(t) T_0 \sqrt{1 - \dot{R}^2(t)/c^2} \tag{6.25}$$

Then, we can obtain the Hamilonian

$$H = \dot{R}\frac{\partial L}{\partial \dot{R}} - L = -\frac{2\pi T_0 R\dot{R}}{2\sqrt{1-\dot{R}^2/c^2}}\Big(-\frac{2\dot{R}}{c^2}\Big) + 2\pi R T_0\sqrt{1-\dot{R}^2/c^2} = \frac{2\pi R(t) T_0}{\sqrt{1-\dot{R}^2(t)/c^2}} \tag{6.26}$$

Acording to the energy onservation law, $\dot{H} = 0$. It gives

$$\frac{d}{dt}\frac{R(t)}{\sqrt{1-\dot{R}^2(t)/c^2}} = 0 \Rightarrow \dot{R}^2(t) - R(t)\ddot{R}(t) = c^2 \Rightarrow R(t) = R\cos\Big(\frac{ct}{R}\Big), \tag{6.27}$$

which has already satisfied the intial conditions: $R(0) = R$ and $\dot{R}(0) = 0$.

6.5 By the definition, it is easy to obtain

$$\mathcal{P}^\sigma_\mu \mathcal{P}^{\sigma\mu} = \frac{T_0^2[c^2 - (\partial_t \boldsymbol{X})^2]}{c^2 - v_\perp^2}\big\{\big[c^2 - (\partial_t\boldsymbol{X})^2\big](\partial_s\boldsymbol{X})^2 - (\partial_s\boldsymbol{X}\cdot\partial_t\boldsymbol{X})^2\big\} = 0 \Rightarrow v = |\partial_t\boldsymbol{X}| = c \tag{6.28}$$

6.6 From Eq. (6.14), we know that

$$\mathcal{L} = -T_0\sqrt{1 - \frac{v_\perp^2}{c^2}}\frac{ds}{d\sigma} \tag{6.29}$$

Then, the canonical momentum density and the Hamiltonian density are given by

$$\boldsymbol{\mathcal{P}}(t,\sigma) = \frac{\partial \mathcal{L}}{\partial(\partial_t\boldsymbol{X})} = -T_0\frac{1}{2}\Big(1-\frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}}\Big[-\frac{1}{c^2}\frac{\partial v_\perp^2}{\partial(\partial_t\boldsymbol{X})}\Big]\frac{ds}{d\sigma} = \frac{T_0}{c^2}\Big(1-\frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}}\frac{ds}{d\sigma}\boldsymbol{v}_\perp \tag{6.30}$$

$$\mathcal{H} = \boldsymbol{\mathcal{P}}(t,\sigma)\cdot\frac{\partial\boldsymbol{X}}{\partial t} - \mathcal{L} = \frac{T_0}{c^2}\Big(1-\frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}}\frac{ds}{d\sigma}v_\perp^2 + T_0\sqrt{1-\frac{v_\perp^2}{c^2}}\frac{ds}{d\sigma} = \frac{T_0}{c^2}\Big(1-\frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}}\frac{ds}{d\sigma} \tag{6.31}$$

The total Hamiltonian can be written as

$$H = \int d\sigma\,\mathcal{H} = \frac{T_0}{c^2}\int ds\Big(1-\frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}} = \mu_0\int ds\Big(1-\frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}}, \tag{6.32}$$

where μ_0 is the rest mass of a string rasing solely from the tension.

6.7 (a) The conditions satisfied by $\mathcal{P}^\sigma_0$, $\mathcal{P}^\sigma_i$, and $\mathcal{P}^\sigma_a$ at the endpoint are stated as follows:

$$\mathcal{P}^\sigma_0(t,0) = 0, \quad \mathcal{P}^\sigma_i(t,0) = 0, \quad \frac{\partial \mathcal{P}^\sigma_a}{\partial\sigma}(t,0) = 0 \tag{6.33}$$

(b) If the string ends on a D0-brane, then the $\sigma = 0$ endpoint is fixed in the spacetime. Therefore, all boundary conditions are automatially satisfied.

(c) For a string ending on a D1-brane, we have

$$\mathcal{P}^\sigma_0 = \frac{T_0}{c}\Big(1-\frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}}\Big(\frac{\partial\boldsymbol{X}}{\partial s}\cdot\frac{\partial\boldsymbol{X}}{\partial t}\Big) = 0 \Rightarrow \boldsymbol{v}\cdot\frac{\partial\boldsymbol{X}}{\partial s} = 0 \tag{6.34}$$

(d)

Chapter 7

String Parameterization and Classical Motion

■ Summary and Supplement

1. Choosing a σ parameterization

$$\frac{\partial \boldsymbol{X}}{\partial \sigma} \cdot \frac{\partial \boldsymbol{X}}{\partial t} = 0, \quad \boldsymbol{v}_\perp = \frac{\partial \boldsymbol{X}}{\partial t} \tag{7.1}$$

$$\dot{X} \cdot X' = 0, \quad X^2 = -c^2 + v_\perp^2, \quad X'^2 = \Big(\frac{ds}{d\sigma}\Big)^2 \tag{7.2}$$

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\Big(1 - \frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}} \frac{ds}{d\sigma}\frac{\partial X^\mu}{\partial t}, \quad \mathcal{P}^{\sigma\mu} = -T_0\Big(1 - \frac{v_\perp^2}{c^2}\Big)^{\frac{1}{2}} \frac{\partial X^\mu}{\partial s} \tag{7.3}$$

$$\frac{\partial}{\partial t}\Big(\frac{T_0}{\sqrt{1 - v_\perp^2/c^2}}\frac{ds}{d\sigma}\Big) = 0, \quad H = \int \frac{T_0 ds}{\sqrt{1 - v_\perp^2/c^2}} \tag{7.4}$$

$$\frac{T_0}{c^2}\frac{1}{\sqrt{1 - v_\perp^2/c^2}}\frac{\partial \boldsymbol{v}_\perp}{\partial t} = \frac{\partial}{\partial s}\Big(T_0\sqrt{1 - \frac{v_\perp^2}{c^2}}\frac{\partial \boldsymbol{X}}{\partial s}\Big), \quad T_{\text{eff}} = T_0\sqrt{1 - \frac{v_\perp^2}{c^2}} \tag{7.5}$$

2. Wave equation and constraints

$$d\sigma' = \frac{ds}{\sqrt{1 - v_\perp^2/c^2}}, \quad \sigma^1 \in [0, \sigma_1], \quad \sigma_1 = \frac{E}{T_0} \tag{7.6}$$

$$\text{Wave equation: } \frac{\partial^2 \boldsymbol{X}}{\partial \sigma^2} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{X}}{\partial t^2} = 0 \tag{7.7}$$

$$\text{Parameterization condition: } \frac{\partial \boldsymbol{X}}{\partial t} \cdot \frac{\partial \boldsymbol{X}}{\partial \sigma} = 0 \tag{7.8}$$

$$\text{Parameterization condition: } \Big(\frac{\partial \boldsymbol{X}}{\partial \sigma}\Big)^2 + \frac{1}{c^2}\Big(\frac{\partial \boldsymbol{X}}{\partial t}\Big)^2 = 1 \tag{7.9}$$

$$\text{Free boundary condition: } \frac{\partial \boldsymbol{X}}{\partial \sigma}\Big|_{\sigma=0} = \frac{\partial \boldsymbol{X}}{\partial \sigma}\Big|_{\sigma=\sigma_1} = 0 \tag{7.10}$$

$$\mathcal{P}^{\tau\mu} = \frac{T_0}{c^2}\frac{\partial X^\mu}{\partial t}, \quad \mathcal{P}^{\sigma\mu} = -T_0\frac{\partial X^\mu}{\partial \sigma} \tag{7.11}$$

3. General motion of an open string

$$\boldsymbol{X}(t, \sigma) = \frac{1}{2}\big[\boldsymbol{F}(ct + \sigma) + \boldsymbol{F}(ct - \sigma)\big], \quad \sigma \in [0, \sigma_1] \tag{7.12}$$

$$\Big|\frac{d\boldsymbol{F}(u)}{du}\Big|^2 = 1 \text{ and } \boldsymbol{F}(u + 2\sigma_1) = \boldsymbol{F}(u) + \frac{2\sigma_1}{c}\boldsymbol{v}_0 \tag{7.13}$$

$$\boldsymbol{F}(u) = \frac{\sigma_1}{\pi}\Big(\cos\frac{\pi u}{\sigma_1}, \sin\frac{\pi u}{\sigma_1}\Big), \quad \boldsymbol{X}(t, \sigma) = \frac{\sigma_1}{\pi}\cos\frac{\pi\sigma}{\sigma_1}\Big(\cos\frac{\pi ct}{\sigma_1}, \sin\frac{\pi ct}{\sigma_1}\Big) \tag{7.14}$$

■ Quick Calculations

7.1 With the periodic conditon of $\boldsymbol{F}(u)$, it is easy to show that

$$\begin{aligned}\boldsymbol{X}(t = t_0 + 2\sigma_2/c, \sigma) &= \frac{1}{2}\big[\boldsymbol{F}(ct_0 + 2\sigma_1 + \sigma) + \boldsymbol{F}(ct_0 + 2\sigma_1 - \sigma)\big] \\ &= \frac{1}{2}\big[\boldsymbol{F}(ct_0 + \sigma) + \boldsymbol{F}(ct_0 - \sigma)\big] + \frac{2\sigma_1}{c}\boldsymbol{v}_0 \\ &= \boldsymbol{X}(t = t_0, \sigma) + \frac{2\sigma_1}{c}\boldsymbol{v}_0\end{aligned} \tag{7.15}$$

Therefore, $\boldsymbol{v}_0$ is the average velocity of any point σ on the string calculated over any time interval of duration $2\sigma_1/c$.

■ Solutions to the Problems

7.1 (d) The length ℓ of an open string parameterized with engery is given by

$$ds = \sqrt{1 - \frac{v_\perp^2}{c^2}}\, d\sigma \Rightarrow \ell = \int_0^{\sigma_1} \sqrt{1 - \frac{v_\perp^2}{c^2}}\, d\sigma \tag{7.16}$$

7.2 (a) For the rotating string, we have $v_\perp = \omega s = 2cs/\ell$. Then, the following holds

$$\mathcal{E}(s) = \frac{dE}{ds} = \frac{T_0}{\sqrt{1 - v_\perp^2/c^2}} = \frac{T_0}{\sqrt{1 - 4s^2/\ell^2}} \tag{7.17}$$

It has singularities at the endpoints $s = \pm\ell/2$. And the total energy is given by

$$E = \int ds\, \mathcal{E}(s) = \int_{-\ell/2}^{\ell/2} ds\, \frac{T_0}{\sqrt{1 - 4s^2/\ell^2}} = \frac{\pi}{2}\ell T_0 \tag{7.18}$$

(b) The average energy density is $\pi T_0/2$, so we have

$$\frac{T_0}{\sqrt{1 - 4s^2/\ell^2}} = \frac{\pi}{2}T_0 \Rightarrow s = \pm\frac{\sqrt{\pi^2 - 4}}{2\pi}\ell \tag{7.19}$$

(c) The energy carried by the string on the interval $[-s, s]$ is given by

$$E = \int_{-s}^{s} dx\, \frac{T_0}{\sqrt{1 - 4x^2/\ell^2}} = \ell T_0 \arcsin\frac{2s}{\ell} \tag{7.20}$$

7.3 (a) The general solution for $\boldsymbol{X}(t, \sigma)$ in terms of a vector function $\boldsymbol{F}(u)$ is given by

$$\boldsymbol{X}(t, \sigma) = \frac{1}{2}\big[\boldsymbol{F}(ct + \sigma) + \boldsymbol{F}(ct - \sigma)\big] \tag{7.21}$$

The following parameterization conditions are required

$$\Big(\frac{\partial \boldsymbol{X}}{\partial \sigma} \pm \frac{1}{c}\frac{\partial \boldsymbol{X}}{\partial t}\Big)^2 = 1 \Rightarrow \Big|\frac{d\boldsymbol{F}(u)}{du}\Big|^2 = 1 \tag{7.22}$$

$$\frac{\partial \boldsymbol{X}}{\partial t}(0, \sigma) = \frac{c}{2}\big[\boldsymbol{F}'(\sigma) + \boldsymbol{F}'(-\sigma)\big] = 0 \Rightarrow \frac{d\boldsymbol{F}(-u)}{du} = -\frac{d\boldsymbol{F}(u)}{du} \Rightarrow \boldsymbol{F}(u) = \boldsymbol{F}(-u) \tag{7.23}$$

(b) We should impose this condition: $\boldsymbol{X}(t, \sigma + \sigma_1) = \boldsymbol{X}(t, \sigma)$, i.e.

$$\begin{aligned}\boldsymbol{F}(ct + \sigma + \sigma_1) + \boldsymbol{F}(ct - \sigma - \sigma_1) &= \boldsymbol{F}(ct + \sigma) + \boldsymbol{F}(ct - \sigma) \\ \Rightarrow \boldsymbol{F}(\sigma + \sigma_1) + \boldsymbol{F}(-\sigma - \sigma_1) &= \boldsymbol{F}(\sigma) + \boldsymbol{F}(-\sigma) \\ \Rightarrow \boldsymbol{F}(\sigma + \sigma_1) &= \boldsymbol{F}(\sigma)\end{aligned} \tag{7.24}$$

(c) $t_{\text{P}} = \sigma_1/c$, $\boldsymbol{X}(t_{\text{P}}, \sigma) = \boldsymbol{X}(0, \sigma)$.

7.4 (a) Using the boundary condition at $\sigma = \sigma_1$, we find

$$\boldsymbol{X}(t,\sigma_1) = \boldsymbol{x}_1 + \frac{1}{2}\big[\boldsymbol{F}(ct+\sigma_1) - \boldsymbol{F}(ct-\sigma_1)\big] = \boldsymbol{x}_2 \Rightarrow \boldsymbol{F}(u+2\sigma_1) - \boldsymbol{F}(u) = 2(\boldsymbol{x}_2 - \boldsymbol{x}_1) \quad (7.25)$$

(b) From the parameterization conditions, we can obtain

$$\Big(\frac{\partial \boldsymbol{X}}{\partial \sigma} \pm \frac{1}{c}\frac{\partial \boldsymbol{X}}{\partial t}\Big)^2 = 1 \Rightarrow \Big|\frac{d\boldsymbol{F}(u)}{du}\Big|^2 = 1 \quad (7.26)$$

(c) According to the information above, we have

$$\begin{aligned}\frac{\partial \boldsymbol{X}}{\partial \sigma}(t,0) &= \boldsymbol{F}'(ct) = (\sin\gamma\cos\omega t, \sin\gamma\sin\omega t, \cos\gamma)\\ &\Rightarrow \boldsymbol{F}'(u) = \Big(\sin\gamma\cos\frac{u\omega}{c}, \sin\gamma\sin\frac{u\omega}{c}, \cos\gamma\Big)\\ &\Rightarrow \boldsymbol{F}(u) = \Big(\frac{c}{\omega}\sin\gamma\sin\frac{u\omega}{c}, -\frac{c}{\omega}\sin\gamma\cos\frac{u\omega}{c}, u\cos\gamma\Big)\end{aligned} \quad (7.27)$$

(d) Using the result in part (a), it is easy to obtain

$$\begin{aligned}&2\Big(\frac{c}{\omega}\sin\gamma\cos\frac{(u+\sigma_1)\omega}{c}\sin\frac{\sigma_1\omega}{c}, \frac{c}{\omega}\sin\gamma\sin\frac{(u+\sigma_1)\omega}{c}\sin\frac{\sigma_1\omega}{c}, \sigma_1\cos\gamma\Big)\\ &= 2(\boldsymbol{x}_2 - \boldsymbol{x}_1) = 2(0,0,L_0) \Rightarrow \omega = \frac{c\pi}{\sigma_1}, \sigma_1 = \frac{L_0}{\cos\gamma}\end{aligned} \quad (7.28)$$

(e) Now, we can directly write down the wave function for this relativistic jumping rope:

$$\boldsymbol{X}(t,\sigma) = \Big(\frac{\sigma_1}{\pi}\sin\gamma\cos\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}, \frac{\sigma_1}{\pi}\sin\gamma\sin\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}, \sigma\cos\gamma\Big) \quad (7.29)$$

(f) First, we calculate the value of $v_\perp^2$:

$$\boldsymbol{v}_\perp = \Big(-c\sin\gamma\sin\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}, c\sin\gamma\cos\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}, 0\Big) \Rightarrow v_\perp^2 = c^2\sin^2\gamma\sin^2\Big(\frac{\pi\sigma}{\sigma_1}\Big) \quad (7.30)$$

Then, the energy distributed in the string is given by

$$\mathcal{E}(z) = T_0\Big(1 - \frac{v_\perp^2}{c^2}\Big)^{-\frac{1}{2}} = T_0\Big[1 - \sin^2\gamma\sin^2\Big(\frac{\pi\sigma}{\sigma_1}\Big)\Big]^{-\frac{1}{2}} = T_0\Big[1 - \sin^2\gamma\sin^2\Big(\frac{\pi z}{L_0}\Big)\Big]^{-\frac{1}{2}} \quad (7.31)$$

7.5 (a) This ansatz is consistent with the constraints on $\boldsymbol{F}(u)$.

(b) It is easy to see that $\boldsymbol{X}'(t,\sigma) = \frac{1}{2}\big[\boldsymbol{F}'(ct+\sigma) + \boldsymbol{F}'(ct-\sigma)\big]$, so we have

$$\boldsymbol{X}'(0,0) = \boldsymbol{F}'(\sigma) \Rightarrow \frac{dy}{d\sigma} = \sin\Big(\gamma\cos\frac{\pi\sigma}{\sigma_1}\Big) \quad (7.32)$$

(c) $\boldsymbol{X}'(t,0) = \boldsymbol{F}'(ct)$.

(d) We will find an integral relation between a, σ_1 and γ:

$$\begin{aligned}\boldsymbol{F}(u+2\sigma) - \boldsymbol{F}(u) &= \Big(\int_u^{u+2\sigma_1}\cos\Big[\gamma\cos\frac{\pi x}{\sigma_1}\Big]dx, \int_u^{u+2\sigma_1}\sin\Big[\gamma\cos\frac{\pi x}{\sigma_1}\Big]dx\Big)\\ &= \Big(\int_0^{2\sigma_1}\cos\Big[\gamma\cos\frac{\pi x}{\sigma_1}\Big]dx, \int_0^{2\sigma_1}\sin\Big[\gamma\cos\frac{\pi x}{\sigma_1}\Big]dx\Big)\\ &= \Big(\int_0^{2\sigma_1}\cos\Big[\gamma\cos\frac{\pi x}{\sigma_1}\Big]dx, 0\Big) \Rightarrow a = \int_0^{\sigma_1}\cos\Big(\gamma\cos\frac{\pi x}{\sigma_1}\Big)dx\end{aligned} \quad (7.33)$$

Assume that γ is small, then the following approxiamtion holds:

$$a \simeq \int_0^{\sigma_1}\Big(1 - \frac{1}{2}\gamma^2\cos^2\frac{\pi x}{\sigma_1}\Big)dx = \sigma_1\Big(1 - \frac{1}{4}\gamma^2\Big) \quad (7.34)$$

(e) Using the following identity

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi}\cos(z\cos\theta)\,d\theta, \quad (7.35)$$

we can obtain

$$a = \int_0^{\sigma_1}\cos\Big(\gamma\cos\frac{\pi x}{\sigma_1}\Big)dx = \frac{\sigma_1}{\pi}\int_0^{\pi}\cos(\gamma\cos\theta)\,d\theta \Rightarrow \frac{a}{\sigma_1} = J_0(\gamma) \quad (7.36)$$

7.6 (a) We only need to examine the components.

$$\begin{aligned}
&\frac{1}{2}\Big\{\cos\Big[\gamma\cos\frac{\pi(ct+\sigma)}{\sigma_1}\Big]+\cos\Big[\gamma\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\Big\}\\
=&\cos\frac{\gamma}{2}\Big[\cos\frac{\pi(ct+\sigma)}{\sigma_1}+\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\cos\frac{\gamma}{2}\Big[\cos\frac{\pi(ct+\sigma)}{\sigma_1}-\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\\
=&\cos\Big(\gamma\cos\frac{\pi ct}{\sigma_1}\cos\frac{\pi\sigma}{\sigma_1}\Big)\cos\Big(\gamma\sin\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}\Big)
\end{aligned}\tag{7.37}$$

$$\begin{aligned}
&\frac{1}{2}\Big\{\sin\Big[\gamma\cos\frac{\pi(ct+\sigma)}{\sigma_1}\Big]+\sin\Big[\gamma\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\Big\}\\
=&\sin\frac{\gamma}{2}\Big[\cos\frac{\pi(ct+\sigma)}{\sigma_1}+\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\cos\frac{\gamma}{2}\Big[\cos\frac{\pi(ct+\sigma)}{\sigma_1}-\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\\
=&\sin\Big(\gamma\cos\frac{\pi ct}{\sigma_1}\cos\frac{\pi\sigma}{\sigma_1}\Big)\cos\Big(\gamma\sin\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}\Big)
\end{aligned}\tag{7.38}$$

When $ct=\sigma_1/2$, the second component is zero. That is to say, the string is horizontal.

(b) It is easy to see that $\dot{\boldsymbol{X}}/c=\frac{1}{2}\big[\boldsymbol{F}'(ct+\sigma)-\boldsymbol{F}'(ct-\sigma)\big]$. Then we have

$$\begin{aligned}
&\frac{1}{2}\Big\{\cos\Big[\gamma\cos\frac{\pi(ct+\sigma)}{\sigma_1}\Big]-\cos\Big[\gamma\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\Big\}\\
=&-\sin\frac{\gamma}{2}\Big[\cos\frac{\pi(ct+\sigma)}{\sigma_1}+\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\sin\frac{\gamma}{2}\Big[\cos\frac{\pi(ct+\sigma)}{\sigma_1}-\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\\
=&-\sin\Big(\gamma\cos\frac{\pi ct}{\sigma_1}\cos\frac{\pi\sigma}{\sigma_1}\Big)\sin\Big(\gamma\sin\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}\Big)
\end{aligned}\tag{7.39}$$

$$\begin{aligned}
&\frac{1}{2}\Big\{\sin\Big[\gamma\cos\frac{\pi(ct+\sigma)}{\sigma_1}\Big]-\sin\Big[\gamma\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\Big\}\\
=&\cos\frac{\gamma}{2}\Big[\cos\frac{\pi(ct+\sigma)}{\sigma_1}+\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\sin\frac{\gamma}{2}\Big[\cos\frac{\pi(ct+\sigma)}{\sigma_1}-\cos\frac{\pi(ct-\sigma)}{\sigma_1}\Big]\\
=&\cos\Big(\gamma\cos\frac{\pi ct}{\sigma_1}\cos\frac{\pi\sigma}{\sigma_1}\Big)\sin\Big(\gamma\sin\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}\Big)
\end{aligned}\tag{7.40}$$

Therefore, the instantaneous transverse velocity satisfies

$$\Big|\frac{1}{c}\frac{\partial\boldsymbol{X}}{\partial t}\Big|=\Big|\sin\Big(\gamma\sin\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}\Big)\Big|\tag{7.41}$$

For $\gamma=\pi/2$, the string midpoint $\sigma=\sigma_1/2$ reaches the speed of lignt when $ct=\sigma_1/2$, meaning that the string is horizontal.

(c) For $\gamma=\sqrt{2}\pi/2$ and $ct=\sigma_1/4$, we have

$$\Big|\sin\Big(\frac{\pi}{\sqrt{2}}\frac{1}{\sqrt{2}}\sin\frac{\pi\sigma}{\sigma_1}\Big)\Big|=1\Rightarrow\sigma=\frac{\sigma_1}{2}\tag{7.42}$$

For $ct=\sigma_1/3$, we have

$$\Big|\sin\Big(\frac{\pi}{\sqrt{2}}\frac{\sqrt{3}}{2}\sin\frac{\pi\sigma}{\sigma_1}\Big)\Big|=1\Rightarrow\sigma=0.174\sigma_1\text{ or }0.826\sigma_1\tag{7.43}$$

Chapter 8

World-sheet Currents

■ Summary and Supplement

1. Conserved currents on the world-sheet

$$S = \int d\xi^0 d\xi^1 \mathcal{L}(\partial_0 X^\mu, \partial_1 X^\mu), \quad (\xi^0, \xi^1) = (\tau, \sigma) \tag{8.1}$$

$$j^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)}, \quad (j^0_\mu, j^1_\mu) = \Big(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \frac{\partial \mathcal{L}}{\partial X^{\mu'}}\Big) = (\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu) \tag{8.2}$$

$$p_\mu(\tau) = \int_0^{\sigma_1} \mathcal{P}^\tau_\mu \, d\sigma, \quad \frac{dp_\mu}{d\tau} = 0, \quad p_\mu = \int_\gamma (\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau) \tag{8.3}$$

2. Lorentz symmetry and associated currents

$$X^\mu \to X^\mu + \delta X^\mu, \quad \delta X^\mu = \epsilon^{\mu\nu} X_\nu, \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \tag{8.4}$$

$$\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - X_\nu \mathcal{P}^\alpha_\mu, \quad \frac{\partial \mathcal{M}^\tau_{\mu\nu}}{\partial \tau} + \frac{\partial \mathcal{M}^\sigma_{\mu\nu}}{\partial \sigma} = 0 \tag{8.5}$$

$$M_{\mu\nu} = \int_\gamma (\mathcal{M}^\tau_{\mu\nu} \, d\sigma - \mathcal{M}^\sigma_{\mu\nu} \, d\tau), \quad M_{\mu\nu} = -M_{\nu\mu} \tag{8.6}$$

3. The slope parameter α'

$$\frac{J}{\hbar} = \alpha' E^2, \quad J = \frac{E^2}{2\pi T_0 c}, \quad \alpha' = \frac{1}{2\pi T_0 \hbar c} \tag{8.7}$$

■ Quick Calculations

8.1 We will use the divergence theorem to prove the result.

$$\frac{dQ_i}{d\xi^0} = \int d\xi^1 d\xi^2 \cdots d\xi^k \frac{\partial j^0_i}{\partial \xi^0} = -\int d\xi^1 d\xi^2 \cdots d\xi^k \Big(\frac{\partial j^1_i}{\partial \xi^1} + \frac{\partial j^2_i}{\partial \xi^2} + \cdots + \frac{\partial j^k_i}{\partial \xi^k}\Big) = -\int_{\partial V} \boldsymbol{j} \cdot d\boldsymbol{A} = 0$$

8.2 For the fixed 2-by-2 matrix A^{ab} that satisfies $A^{ab} v_a v_b = 0$, we have

$$A^{11} v_1 v_1 + A^{12} v_1 v_2 + A^{21} v_2 v_1 + A^{22} v_2 v_2 = 0 \Rightarrow A^{11} = 0, A^{12} = -A^{21}, A^{22} = 0 \tag{8.8}$$

8.3 For a 4-by-4 matrix $\epsilon^{\mu\nu}$ that satisfies $\epsilon^{\mu\nu} v_\mu v_\nu = 0$, the conclusion is the same: $\epsilon^{\mu\nu}$ must be antisymmetric.

8.4 $\epsilon_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}\epsilon^{\alpha\beta} = -\eta_{\mu\alpha}\eta_{\nu\beta}\epsilon^{\beta\alpha} = -\eta_{\alpha\mu}\eta_{\beta\nu}\epsilon^{\beta\alpha} = -\epsilon_{\nu\mu}$.

8.5 For the boost with very small β, we have

$$x'^0 - x^0 = (\gamma - 1)x^0 - \gamma\beta x^1 \simeq -\beta x_1 \Rightarrow \epsilon^{01} = -\beta \tag{8.9}$$

$$x'^1 - x^1 = -\gamma\beta x^0 + (\gamma - 1)x^1 \simeq \beta x_0 \Rightarrow \epsilon^{10} = \beta \tag{8.10}$$

And all other values are zero.

8.5 $J = I\omega$, $E = \frac{1}{2}I\omega^2 \Rightarrow J \sim \sqrt{E}$.

8.6 $[\alpha'] = [E]^{-2}$, $[\hbar] = [E]T$, $[c] = LT^{-1} \Rightarrow [\ell_s] = L$.

■ Solutions to the Problems

8.1 (a) The variation can be written as $\delta\boldsymbol{q}(t) = \epsilon\boldsymbol{n}\times\boldsymbol{q}$, where ϵ is an infinitesimal constant and $\boldsymbol{n}$ is the rotation axis. Then we have

$$\dot{\boldsymbol{q}}' = \dot{\boldsymbol{q}} + \epsilon\boldsymbol{n}\times\boldsymbol{q} \Rightarrow \dot{q}'^2 = \dot{q}^2 + \epsilon^2(\boldsymbol{n}\times\dot{\boldsymbol{q}})^2 \simeq \dot{q}^2 \tag{8.11}$$

The Lagrangian only depends on $\dot{q}^2$, therefore L is invariant.

(b) The conserved charge associated with this symmetry transformation is given by

$$\epsilon Q = \frac{\partial L}{\partial\dot{\boldsymbol{q}}}\cdot\delta\boldsymbol{q} = \boldsymbol{p}\cdot(\boldsymbol{n}\times\boldsymbol{q}) = \boldsymbol{n}\cdot(\boldsymbol{q}\times\boldsymbol{p}) = \boldsymbol{n}\cdot\boldsymbol{J}, \tag{8.12}$$

where $\boldsymbol{J}$ is the vector angular momentum.

8.2 (a) The variation of coordinates and the conserved charges are given by

$$\delta q^i(t) = \epsilon^i(t)h(\boldsymbol{q}(t);t), \quad \epsilon^i Q_i = \frac{\partial L}{\partial\dot{q}^a}\delta q^a \tag{8.13}$$

Considering the Euler-Lagrange equations, we can obtain

$$\epsilon^i\frac{dQ_i}{dt} = \frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{q}^a}\Big)\delta q^a + \frac{\partial L}{\partial\dot{q}^a}\frac{d}{dt}(\delta q^a) = \frac{\partial L}{\partial q^a}\delta q^a + \frac{\partial L}{\partial\dot{q}^a}\frac{d}{dt}(\delta q) = 0 \tag{8.14}$$

(b) For a world with no spatial dimentions, $\alpha = 0$. And the corresponding equations become

$$\epsilon^i j_i^0 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^a}\delta\phi^a, \quad \frac{d}{dt}j_i^0 = 0, \quad Q_i = cq_i \tag{8.15}$$

8.3

8.4 (a) $T_0 = 8.5\times10^{14}\,\text{GeV}\cdot\text{m}^{-1}$, $\ell_s = 1.92\times10^{-14}\,\text{cm}$.

(b) $\alpha' = 2.58\times10^{-33}\,\text{GeV}^{-2}$, $T_0 = 3\times10^{47}\,\text{GeV}\cdot\text{m}^{-1}$.

8.5 For the relativistic jumping rope, we have

$$\boldsymbol{X}(t,\sigma) = \Big(\frac{\sigma_1}{\pi}\sin\gamma\cos\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}, \frac{\sigma_1}{\pi}\sin\gamma\sin\frac{\pi ct}{\sigma_1}\sin\frac{\pi\sigma}{\sigma_1}, \sigma\cos\gamma\Big) \tag{8.16}$$

$$\boldsymbol{\mathcal{P}}^\tau = \frac{T_0}{c^2}\frac{\partial\boldsymbol{X}}{\partial t} = \frac{T_0}{c}\sin\gamma\sin\frac{\pi\sigma}{\sigma_1}\Big(-\sin\frac{\pi ct}{\sigma_1}, \cos\frac{\pi ct}{\sigma_1}, 0\Big) \tag{8.17}$$

Then, the z-component of angular momentum is given by

$$J_z = M_{12} = \int_0^{\sigma_1}(X_1\mathcal{P}_2^\tau - X_2\mathcal{P}_1^\tau)\,d\sigma = \frac{T_0\sigma_1}{\pi c}\sin^2\gamma\int_0^{\sigma_1}\sin^2\frac{\pi\sigma}{\sigma_1}\,d\sigma = \frac{T_0\sigma_1^2}{2\pi c}\sin^2\gamma \tag{8.18}$$

Since $\sigma_1 = E/T_0$, we have found

$$J_z = \frac{E^2}{2\pi T_0 c}\sin^2\gamma \Rightarrow \frac{J_z}{\hbar} = (\sin^2\gamma)\alpha' E^2 \tag{8.19}$$

8.6 With the Euler-Lagrange equation, we can show that

$$\epsilon\frac{dQ}{dt} = \frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{q}}\Big)\delta q + \frac{\partial L}{\partial\dot{q}}\frac{d}{dt}(\delta q) - \epsilon\frac{d\Lambda}{dt} = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial\dot{q}}\delta\dot{q} - \delta L = 0 \tag{8.20}$$

For the transformation $q(t) \to q(t) + \epsilon\dot{q}(t)$, we have

$$\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial\dot{q}}\delta\dot{q} = \epsilon\Big[\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{q}}\Big)\dot{q} + \frac{\partial L}{\partial\dot{q}}\ddot{q}\Big] = \epsilon\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{q}}\dot{q}\Big) \Rightarrow \Lambda = \frac{\partial L}{\partial\dot{q}}\dot{q} \tag{8.21}$$

$$\epsilon Q = \frac{\partial L}{\partial\dot{q}}\delta q - \epsilon\Lambda = \frac{\partial L}{\partial\dot{q}}\dot{q} - \epsilon\Lambda \Rightarrow Q = 0 \tag{8.22}$$

8.7 The proof is very similar to that in Eq. (8.20):

$$\begin{aligned}\epsilon^i \frac{\partial j_i^\alpha}{\partial \xi^\alpha} &= \frac{\partial}{\partial \xi^\alpha}\Big[\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\Big]\delta\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\frac{\partial \delta\phi^a}{\partial \xi^\alpha} - \epsilon^i \frac{\partial \Lambda_i^\alpha}{\partial \xi^\alpha} \\ &= \frac{\partial \mathcal{L}}{\partial \phi^a}\delta\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\delta(\partial_\alpha \phi^a) - \delta\mathcal{L} = 0\end{aligned} \tag{8.23}$$

For the transformation $\phi^a(\xi^\beta) \to \phi^a(\xi^\beta) + \epsilon^\beta \partial_\beta \phi^a$, we have

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi^a}\delta\phi^a + \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\delta(\partial_\alpha \phi^a) = \epsilon^\beta\Big[\frac{\partial}{\partial \xi^\alpha}\Big(\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\Big)\partial_\beta \phi^a + \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\partial_\alpha \partial_\beta \phi^a\Big] \\ &= \epsilon^\beta \frac{\partial}{\partial \xi^\alpha}\Big(\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\partial_\beta \phi^a\Big) \Rightarrow \Lambda_\beta^\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\partial_\beta \phi^a = \delta_\beta^\alpha \mathcal{L}\end{aligned} \tag{8.24}$$

$$\epsilon^\beta j_\beta^\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\delta\phi^a - \epsilon^\beta \Lambda_\beta^\alpha = \epsilon^\beta\Big(\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\partial_\beta \phi^a - \delta_\beta^\alpha \mathcal{L}\Big) \Rightarrow j_\beta^\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^a}\partial_\beta \phi^a - \delta_\beta^\alpha \mathcal{L} \tag{8.25}$$

It is easy to see that $j_0^0 = \mathcal{H}$, i.e. the Hamiltonian density.

Chapter 9

Light-cone Relativistic Strings

■ Summary and Supplement

1. The σ parameterization

$$X^0(\tau,\sigma)=c\tau, \quad n_\mu X^\mu(\tau,\sigma)=\lambda\tau \tag{9.1}$$

$$\hbar=c=1, \quad L=T,\ M=L^{-1}, \quad \ell_s=\sqrt{\alpha'} \tag{9.2}$$

$$S=-\frac{1}{2\pi\alpha'}\int_{\tau_i}^{\tau_f}d\tau\int_0^{\sigma_1}d\sigma\sqrt{(\dot{X}\cdot X')^2-(\dot{X})^2(X')^2} \tag{9.3}$$

$$\text{open strings: } (n\cdot p)\sigma=\pi\int_0^\sigma d\tilde{\sigma}\, n\cdot\mathcal{P}^\tau(\tau,\tilde{\sigma}), \quad n\cdot X(\tau,\sigma)=2\alpha'(n\cdot p)\tau \tag{9.4}$$

$$\text{closed strings: } (n\cdot p)\sigma=2\pi\int_0^\sigma d\tilde{\sigma}\, n\cdot\mathcal{P}^\tau(\tau,\tilde{\sigma}), \quad n\cdot X(\tau,\sigma)=\alpha'(n\cdot p)\tau \tag{9.5}$$

2. Constraints and wave equations

$$\dot{X}\cdot X'=0, \quad \dot{X}^2+X'^2=0, \quad (\dot{X}\pm X')^2=0 \tag{9.6}$$

$$\mathcal{P}^{\tau\mu}=\frac{1}{2\pi\alpha'}\dot{X}^\mu, \quad \mathcal{P}^{\sigma\mu}=-\frac{1}{2\pi\alpha'}X^{\mu\prime}, \quad \ddot{X}^\mu-X^{\mu\prime\prime}=0 \tag{9.7}$$

3. Wave equation and mode expansions

$$X^\mu(\tau,\sigma)=f_0^\mu+f_1^\mu\tau+\sum_{n=1}^\infty(A_n^\mu\cos n\tau+B_n^\mu\sin n\tau)\cos n\sigma \tag{9.8}$$

$$A_n^\mu\cos n\tau+B_n^\mu\sin n\tau=-i\sqrt{2\alpha'/n}\,(a_n^{\mu*}e^{in\tau}-a_n^\mu e^{-in\tau}), \quad f_1^\mu=2\alpha' p^\mu \tag{9.9}$$

$$X^\mu(\tau,\sigma)=x_0^\mu+2\alpha' p^\mu\tau-i\sqrt{2\alpha'}\sum_{n=1}^\infty(a_n^{\mu*}e^{in\tau}-a_n^\mu e^{-in\tau})\frac{\cos n\sigma}{\sqrt{n}} \tag{9.10}$$

$$\alpha_0^\mu=\sqrt{2\alpha'}p^\mu, \quad \alpha_n^\mu=a_n^\mu\sqrt{n}, \quad \alpha_{-n}^\mu=a_n^{\mu*}\sqrt{n}, \quad n\geq 1 \tag{9.11}$$

$$X^\mu(\tau,\sigma)=x_0^\mu+\sqrt{2\alpha'}\alpha_0^\mu\tau+i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma \tag{9.12}$$

$$\dot{X}^\mu\pm X^{\mu\prime}=\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in(\tau\pm\sigma)} \tag{9.13}$$

4. Light-cone solution of equations of motion

$$n_\mu=\Big(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,\ldots,0\Big), \quad n\cdot X=X^+, \quad n\cdot p=p^+ \tag{9.14}$$

$$X^+(\tau,\sigma)=\beta\alpha' p^+\tau, \quad p^+\sigma=\frac{2\pi}{\beta}\int_0^\sigma d\tilde{\sigma}\, p^{\tau+}(\tau,\sigma), \quad \beta=2,1 \tag{9.15}$$

$$X^I=(X^2,X^3,\ldots,X^d), \quad \dot{X}^-\pm X^{-\prime}=(2\beta\alpha' p^+)^{-1}(\dot{X}^I\pm X^{I\prime})^2 \tag{9.16}$$

$$x_0^+ = 0, \quad \alpha_n^+ = \alpha_{-n}^+ = 0, \quad n = 1, 2, \ldots, \infty \tag{9.17}$$

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+}L_n^\perp, \quad L_n^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{n-p}^I \alpha_p^I, \quad 2p^+p^- = \frac{1}{\alpha'}L_0^\perp \tag{9.18}$$

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{p^+}\sum_{n\in\mathbb{Z}} L_n^\perp e^{-in(\tau\pm\sigma)} = \frac{1}{4\alpha' p^+}(\dot{X}^- \pm X^{-\prime})^2 \tag{9.19}$$

$$M^2 = 2p^+p^- - p^Ip^I = \frac{1}{\alpha'}\sum_{n=1}^{\infty} n a_n^{I*} a_n^I \tag{9.20}$$

■ Quick Calculations

9.1 $e = \hbar c/\ell = 2\times 10^{-5}\times 10^{18} = 2\times 10^{13}\,\text{eV} = 20\,\text{TeV}$.

9.2 Since $t^\mu t_\mu < 0$, then we can prove the vector v^μ is timelike:

$$v^\mu v_\mu = \Big(t^\mu - \frac{t\cdot X'}{X'\cdot X'}X^{\mu\prime}\Big)\Big(t_\mu - \frac{t\cdot X'}{X'\cdot X'}X'_\mu\Big) = t^\mu t_\mu - \frac{(t\cdot X')^2}{X'\cdot X'} < 0 \tag{9.21}$$

9.3 It is obvious that $X^\mu(\tau,\sigma)$ is real.

9.4 Using the relation $\sqrt{2\alpha'}\alpha_0^- = L_n^\perp/p^+$, we can rewrite the expression as

$$\begin{aligned} X^-(\tau,\sigma) &= x_0^- + \sqrt{2\alpha'}\alpha_0^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\tau}\cos n\sigma \\ &= x_0^- + \frac{1}{p^+}L_0^\perp\tau + \frac{i}{p^+}\sum_{n\neq 0}\frac{1}{n}L_n^\perp e^{-in\tau}\cos n\sigma \end{aligned} \tag{9.22}$$

9.5 When all a_n^I vanish, we have $L_n^\perp = 0$ for $n \geq 1$. Therefore, $X^-(\tau,\sigma) = x_0^- + \sqrt{2\alpha'}\alpha_0^-\tau$.

■ Solutions to the Problems

9.1 (a) $(n_\mu + b_\mu)(n^\mu + b^\mu) = b_\mu b^\mu \geq 0$. Therefore, b^μ is spacelike of null.

(b) If b^μ is null, we have $b_\mu(n^\mu - \lambda b^\mu) = 0$ and $\lambda n_\mu)(n^\mu - \lambda b^\mu) = 0$. Then $(b_\mu - \lambda n_\mu)(n^\mu - \lambda b^\mu) = 0$. Therefore, $b^\mu = \lambda n^\mu$.

(c) The set of vectors b^μ that satisfies $n_\mu b^\mu = 0$ is a hyperplane orthogonal to the vector n^μ. Therefore, it is a space of dimension (D_1). And the subset of null vectors forms a subspace of dimension one.

(d) For $D = 2$ and $n \cdot X = X^+$, we can obtain

$$n_\mu = \Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big), \quad n^\mu = \Big(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big) \tag{9.23}$$

9.2 (a) Using Eq. (9.16), we can find

$$\partial_\tau X^- = \frac{(\partial_\tau X^I)^2 + (\partial_\sigma X^I)^2}{2\beta\alpha' p^+}, \quad \partial_\sigma X^- = \frac{(\partial_\tau X^I)(\partial_\sigma X^I)}{\beta\alpha' p^+} \tag{9.24}$$

From the consistency condtion $\partial_\sigma\partial_\tau X^- = \partial_\tau\partial_\sigma X^-$, it is easy to prove that

$$\partial_\sigma\big[(\partial_\tau X^I)^2 + (\partial_\sigma X^I)^2\big] = 2\partial_\tau\big[(\partial_\tau X^I)(\partial_\sigma X^I)\big] \Rightarrow \partial_\tau^2 X^I = \partial_\sigma^2 X^I \tag{9.25}$$

(b) If the transverse coordinates X^I satisfy the wave function, then it follows

$$\partial_\tau^2 X^- - \partial_\sigma^2 X^- = (\beta\alpha' p^+)^{-1}(\partial_\tau^2 X^I)(\partial_\tau^2 X^I - \partial_\sigma^2 X^I) = 0 \tag{9.26}$$

(c) The coordinates satisfying Neumann boundary conditions means that $\partial_\sigma X^I = 0$ at $\sigma = 0, \pi$, while satisfing Dirichlet boundary conditions indicates that $\partial_\tau X^I = 0$. Since $\partial_\sigma X^-$ is determined by the product of $\partial_\tau X^I$ and $\partial_\sigma X^I$, it is always zero. Therefore, X^- will always satisfy Neumann boundary conditions.

9.3 (a) $M^2 = (\alpha')^{-1} \sum_{n=1}^{\infty} \alpha_n^{\mu} \alpha_{-n}^{\mu} = (\alpha')^{-1}(a^2 + a^2) = 2a^2/\alpha'$.

(b) The length of the string is given by $l = 2\pi M\alpha' = 2\pi a\sqrt{2\alpha'}$. And the explicit functions of $X^{(2)}(\tau,\sigma)$ and $X^{(3)}(\tau,\sigma)$ are given by

$$X^{(2)}(\tau,\sigma) = x_0^{(2)} + i\sqrt{2\alpha'}a(e^{-i\tau} - e^{i\tau})\cos\sigma = 2\sqrt{2\alpha'}a\sin\tau\cos\sigma \tag{9.27}$$

$$X^{(3)}(\tau,\sigma) = x_0^{(3)} + i\sqrt{2\alpha'}ia(e^{-i\tau} + e^{i\tau})\cos\sigma = -2\sqrt{2\alpha'}a\cos\tau\cos\sigma \tag{9.28}$$

(c) $L_0^{\perp} = \frac{1}{2}\left[\alpha_{-1}^{(2)}\alpha_1^{(2)} + \alpha_{-1}^{(3)}\alpha_1^{(3)}\right] = a^2$, $L_0^{\perp} = 0$ $(n \neq 0)$. $X^{-}(\tau,\sigma) = a^2\tau/p^{+}$.

(d) $\tau = \omega t = \pi t/l$. The value of p^+ can be determined by the following

$$X^1(\tau,\sigma) = \frac{1}{\sqrt{2}}(X^+ - X^-) = (\sqrt{2}p^+\alpha' - \frac{a^2}{\sqrt{2}p^+})\tau = 0 \Rightarrow p^+ = \frac{a}{\sqrt{2\alpha'}} \tag{9.29}$$

9.4 (a)

$$X^2(\tau,\sigma) = x_0^2 + i\sqrt{2\alpha'}a(e^{-i\tau} - e^{i\tau})\cos\sigma = 2\sqrt{2\alpha'}a\sin\tau\cos\sigma \tag{9.30}$$

Chapter 10

Light-cone Fields and Particles

■ Summary and Supplement

1. Classical scalar fields

$$S=\int d^Dx\Big(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m^2\phi^2\Big),\quad (\partial^2-m^2)\phi=0 \tag{10.1}$$

$$\phi(t,\boldsymbol{x})=ae^{-iE_pt+i\boldsymbol{p}\cdot\boldsymbol{x}}+a^*e^{iE_pt-i\boldsymbol{p}\cdot\boldsymbol{x}},\quad E_p=\sqrt{p^2+m^2} \tag{10.2}$$

$$\phi(x)=\int\frac{d^Dp}{(2\pi)^D}\,e^{ip\cdot x}\phi(p),\quad (p^2-m^2)\phi(p)=0 \tag{10.3}$$

$$\boldsymbol{x}_T=(x^2,x^3,\ldots,x^d),\quad \Big(-2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-}+\frac{\partial}{\partial x^I}\frac{\partial}{\partial x^I}-m^2\Big)\phi(x^+,x^-,\boldsymbol{x}_T)=0 \tag{10.4}$$

$$\boldsymbol{p}_T=(p^2,p^3,\ldots,p^d),\quad \Big[i\frac{\partial}{\partial x^+}-\frac{1}{2p^+}\big(p^Ip^I+m^2\big)\Big]\phi(x^+,p^+,\boldsymbol{p}_T)=0 \tag{10.5}$$

$$x^+=\frac{p^+\tau}{m^2},\quad \Big[i\frac{\partial}{\partial\tau}-\frac{1}{2m^2}\big(p^Ip^I+m^2\big)\Big]\phi(\tau,p^+,\boldsymbol{p}_T)=0 \tag{10.6}$$

$$\big[a_p,a_p^\dagger\big]=1,\quad \big[a_{-p},a_{-p}^\dagger\big]=1,\quad \big[a(t),\dot{a}^\dagger(t)\big]=\big[a^\dagger(t),\dot{a}(t)\big]=2iE_p \tag{10.7}$$

$$H=E_p\big(a_p^\dagger a_p+a_{-p}^\dagger a_{-p}\big),\quad \boldsymbol{P}=\boldsymbol{p}\big(a_p^\dagger a_p-a_{-p}^\dagger a_{-p}\big) \tag{10.8}$$

2. Maxwell fields

$$\partial^2A^\mu-\partial^\mu(\partial\cdot A)=0,\quad p^2A^\mu-p^\mu(p\cdot A)=0 \tag{10.9}$$

$$A^\mu(p)\to A^\mu(p)+ip^\mu\epsilon(p),\quad A^+(p)=0 \tag{10.10}$$

$$p\cdot A=0,\quad p^+A^-=p^IA^I,\quad p^2A^I=0 \tag{10.11}$$

3. Gravitational fields

$$p^2h^{\mu\nu}-p_\alpha(p^\mu h^{\nu\alpha}+p^\nu h^{\mu\alpha}+p^\mu p^\nu h=0 \tag{10.12}$$

$$\delta h^{\mu\nu}(p)=ip^\mu\epsilon^\nu(p)+ip^\nu\epsilon^\mu(p),\quad \delta h=2ip\cdot\epsilon \tag{10.13}$$

$$h^{++}=h^{+-}=h^{+I}=0,\quad h=0,\ h^{II}=0,\quad p^2h^{\mu\nu}=0 \tag{10.14}$$

$$p^+h^{I-}=p_Jh^{IJ},\quad p^+h^{--}=p_Ih^{-I},\quad p^2h^{IJ}=0 \tag{10.15}$$

■ Quick Calculations

10.1 By the definition, we have

$$\mathcal{H}=\Pi\partial_0\phi-\mathcal{L}=\Pi^2-\Big[\frac{1}{2}\Pi^2-\frac{1}{2}(\nabla\phi)^2-\frac{1}{2}m^2\phi^2\Big]=\frac{1}{2}\Pi^2+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2 \tag{10.16}$$

10.2 With the mass-shell condition, we have

$$p^2+m^2=-2p^+p^-+p^Ip^I+m^2=0\Rightarrow p^-=\frac{1}{2p^+}\big(p^Ip^I+m^2\big) \tag{10.17}$$

10.3 $\partial_0\phi_p$ and $\nabla\phi_p$ are given by

$$\partial_0\phi_p = \frac{1}{\sqrt{2VE_p}}\big[\dot{a}(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + \dot{a}^*(t)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\big], \quad \nabla\phi_p = \frac{i}{\sqrt{2VE_p}}\big[a(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^*(t)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\big]\boldsymbol{p} \tag{10.18}$$

Using the quantization condition, we can integrate the action as

$$\begin{aligned} S &= \int dt \int d^d x \left[\frac{1}{2VE_p}\dot{a}(t)\dot{a}^*(t) - \frac{1}{2VE_p}a(t)a^*(t)p^2 - \frac{1}{2VE_p}a(t)a^*(t)m^2\right] \\ &= \int dt \int d^d x \, \frac{1}{V}\left[\frac{1}{2E_p}\dot{a}(t)\dot{a}^*(t) - \frac{1}{2}E_p a(t)a^*(t)\right] \\ &= \int dt \left[\frac{1}{2E_p}\dot{a}(t)\dot{a}^*(t) - \frac{1}{2}E_p a(t)a^*(t)\right] \end{aligned} \tag{10.19}$$

10.4 With the quantization condition, it is easy to obtain

$$\begin{aligned} H &= \int d^d x \left[\frac{1}{2VE_p}\dot{a}(t)\dot{a}^*(t) + \frac{1}{2VE_p}a(t)a^*(t)p^2 + \frac{1}{2VE_p}a(t)a^*(t)m^2\right] \\ &= \frac{1}{2E_p}\dot{a}(t)\dot{a}^*(t) + \frac{1}{2}E_p a(t)a^*(t) \end{aligned} \tag{10.20}$$

10.5 Using the relations $\big[a(t), \dot{a}^\dagger(t)\big] = \big[a^\dagger(t), \dot{a}(t)\big] = 2iE_p$, we can easily check that

$$[q_2(t), p_2(t)] = -\frac{1}{4E_p}\big[a(t) - a^\dagger(t), \dot{a}(t) - \dot{a}^\dagger(t)\big] = -\frac{1}{4E_p}(-2iE_p - 2iE_p) = i \tag{10.21}$$

$$[q_1(t), p_2(t)] = \frac{1}{4iE_p}\big[a(t) + a^\dagger(t), \dot{a}(t) - \dot{a}^\dagger(t)\big] = \frac{1}{4iE_p}(2iE_p - 2iE_p) = 0 \tag{10.22}$$

10.6 Using the facts $\big[a_p, a_k^\dagger\big] = \delta_{p,k}$ and $\big[a_p^\dagger, a_k^\dagger\big] = 0$, we can obtain

$$\begin{aligned} \boldsymbol{P}a_{p_1}^\dagger a_{p_2}^\dagger \cdots a_{p_n}^\dagger |\Omega\rangle &= \sum_{\boldsymbol{k}} \boldsymbol{k} a_k^\dagger (a_{p_1}^\dagger a_k + \delta_{p_1,k}) a_{p_2}^\dagger \cdots a_{p_n}^\dagger |\Omega\rangle \\ &= \sum_{\boldsymbol{k}} \boldsymbol{k} a_{p_1}^\dagger a_k^\dagger (a_{p_2}^\dagger a_k + \delta_{p_2,k}) \cdots a_{p_n}^\dagger |\Omega\rangle + \boldsymbol{p}_1 a_{p_1}^\dagger a_{p_2}^\dagger \cdots a_{p_n}^\dagger |\Omega\rangle \\ &= \sum_{n=1}^{k} \boldsymbol{p}_n a_{p_1}^\dagger a_{p_2}^\dagger \cdots a_{p_n}^\dagger |\Omega\rangle \end{aligned} \tag{10.23}$$

Similarly, we can prove that $H a_{p_1}^\dagger a_{p_2}^\dagger \cdots a_{p_n}^\dagger |\Omega\rangle = \sum_{n=1}^{k} E_n a_{p_1}^\dagger a_{p_2}^\dagger \cdots a_{p_n}^\dagger |\Omega\rangle$.

10.7 We only need to prove that $N a_{p_1}^\dagger a_{p_2}^\dagger \cdots a_{p_n}^\dagger |\Omega\rangle = n a_{p_1}^\dagger a_{p_2}^\dagger \cdots a_{p_n}^\dagger |\Omega\rangle$.

■ Solutions to the Problems

10.1 Using the results in Eq. (10.18) and the quantizition condition

$$\int_0^{L_1} dx^1 \cdots \int_0^{L_d} dx^d \, \exp(\pm 2i\boldsymbol{p}\cdot\boldsymbol{x}) = 0, \tag{10.24}$$

we can evaluate the integral and obtain

$$\begin{aligned} \boldsymbol{P} &= -\frac{i\boldsymbol{p}}{2VE_p}\int d^d x \big[\dot{a}(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + \dot{a}^*(t)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\big]\big[a(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^*(t)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\big] \\ &= -\frac{i\boldsymbol{p}}{2VE_p}\int d^d x \big[\dot{a}^*(t)a(t) - \dot{a}(t)a^*(t)\big] \\ &= -\frac{i\boldsymbol{p}}{2E_p}(\dot{a}^* a - \dot{a}a^*) \end{aligned} \tag{10.25}$$

Then, with the exoression of $a(t)$, it is easy to get

$$\begin{aligned} \boldsymbol{P} = &-\frac{i\boldsymbol{p}}{2E_p}(iE_p)\big[\big(a_p^* e^{iE_pt} - a_{-p}e^{-iE_pt}\big)\big(a_p e^{-iE_pt} + a_{-p}^* e^{iE_pt}\big) \\ &- \big(-a_p e^{-iE_pt} + a_{-p}^* e^{iE_pt}\big)\big(a_p^* e^{iE_pt} + a_{-p}e^{-iE_pt}\big)\big] \\ = &\, \boldsymbol{p}(a_p^* a_p - a_{-p}^* a_{-p}) \end{aligned} \tag{10.26}$$

10.2 (a) Using the quantizition condition, we can prove that

$$\frac{1}{V}\int d\boldsymbol{x}'\, f(\boldsymbol{x}')e^{-i\boldsymbol{p}\cdot\boldsymbol{x}'} = \frac{1}{V}\sum_{\boldsymbol{p}'}\int d\boldsymbol{x}'\, f(\boldsymbol{p}')e^{i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{x}'} = f(\boldsymbol{p}) \tag{10.27}$$

Plug this back into the Fourier series, we can obtain a representation for the delta function:

$$\int d\boldsymbol{x}'\, f(\boldsymbol{x}')\delta^d(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{V}\sum_{\boldsymbol{p}}\int d\boldsymbol{x}'\, f(\boldsymbol{x}')e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \Rightarrow \delta^d(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{V}\sum_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \tag{10.28}$$

(b) For the complete scale field expansion, we have

$$\Pi(t,\boldsymbol{x}) = \partial_0\phi(t,\boldsymbol{x}) = \frac{i}{\sqrt{2V}}\sum_{\boldsymbol{p}}\sqrt{E_p}\left(a_p^\dagger e^{iE_pt-i\boldsymbol{p}\cdot\boldsymbol{x}} - a_p e^{-iE_pt+i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{10.29}$$

Then, it is very easy to show that

$$\left[\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x}')\right] = i\delta^d(\boldsymbol{x}-\boldsymbol{x}') \tag{10.30}$$

10.3 (a) For the Lorentz covariant equation $A^\mu = B^\mu$, $\mu = 0,1,\ldots,d$, we have

$$A^+ = \frac{1}{\sqrt{2}}(A^0+A^1) = \frac{1}{\sqrt{2}}(B^0+B^1) = B^+ \tag{10.31}$$

$$A^- = \frac{1}{\sqrt{2}}(A^0-A^1) = \frac{1}{\sqrt{2}}(B^0-B^1) = B^- \tag{10.32}$$

It is obvious that $A^I = B^I$ for $I = 2,\ldots,d$.

(b) By the definition, we can obtain

$$R^{++} = A^+B^+ = \frac{1}{\sqrt{2}}(A^0+A^1)\frac{1}{\sqrt{2}}(B^0+B^1) = \frac{1}{2}(R^{00}+R^{01}+R^{10}+R^{11}) \tag{10.33}$$

$$R^{+-} = A^+B^- = \frac{1}{\sqrt{2}}(A^0+A^1)\frac{1}{\sqrt{2}}(B^0-B^1) = \frac{1}{2}(R^{00}-R^{01}+R^{10}-R^{11}) \tag{10.34}$$

$$R^{-+} = A^-B^+ = \frac{1}{\sqrt{2}}(A^0-A^1)\frac{1}{\sqrt{2}}(B^0+B^1) = \frac{1}{2}(R^{00}+R^{01}-R^{10}-R^{11}) \tag{10.35}$$

$$R^{--} = A^-B^- = \frac{1}{\sqrt{2}}(A^0-A^1)\frac{1}{\sqrt{2}}(B^0-B^1) = \frac{1}{2}(R^{00}-R^{01}-R^{10}+R^{11}) \tag{10.36}$$

Then, it is easy to see that an equaltiy $R^{\mu\nu} = S^{\mu\nu}$ between Lorentz tensors implies the equality of the light-cone components.

(c) For the Minkowski metric $\eta = \text{diag}(-1,1,1,1)$, the relations above become

$$\eta^{++} = \eta^{--} = 0, \quad \eta^{+-} = \eta^{-+} = -1 \tag{10.37}$$

(d) For the antisymmetric electromagnetic field strength

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}, \tag{10.38}$$

we can calculate the light-cone components

$$F^{++} = F^{--} = 0, \quad F^{+-} = -E_x, \quad F^{-+} = E_x \tag{10.39}$$

$$F^{+2} = \frac{1}{\sqrt{2}}(E_y+B_z), \quad F^{+3} = \frac{1}{\sqrt{2}}(E_z-B_y) \tag{10.40}$$

$$F^{-2} = \frac{1}{\sqrt{2}}(E_y-B_z), \quad F^{-3} = \frac{1}{\sqrt{2}}(E_z+B_y) \tag{10.41}$$

$$F^{22} = F^{33} = 0, \quad F^{23} = B_x, \quad F^{32} = -B_x \tag{10.42}$$

10.4 For the uniform constant electric field $\boldsymbol{E} = E_0 \boldsymbol{e}_x$, we can choose

$$A^\mu = (-E_0 x, E_0 x, 0, 0), \tag{10.43}$$

which automatically satisfies the condition $A^+ = 0$. Then, we have

$$A^- = \frac{2}{\sqrt{2}}(A^0 - A^1) = -\sqrt{2}E_0 x = E_0(x^- - x^+), \quad A^I = 0 \tag{10.44}$$

10.5 A pure gauge of a gravitational field is defined as $h^{\mu\nu}(p) = ip^\mu \epsilon^\nu(p) + ip^\nu \epsilon^\mu(p)$. Then, from the equation of the field

$$p^2 h^{\mu\nu} = p_\alpha(p^\mu h^{\nu\alpha} + p^\nu h^{\mu\alpha}) \Rightarrow h^{\mu\nu} = i\frac{(-i)(p^\mu p_\alpha h^{\nu\alpha} + p^\nu p_\alpha h^{\mu\alpha})}{p^2} \tag{10.45}$$

we can see that: if $p^2 \neq 0$, setting $\epsilon^\mu = p_\alpha h^{\mu\alpha}$ and $\epsilon^\nu = p_\alpha h^{\nu\alpha}$ will just yield a pure gauge.

10.6 (a) Using the antisymmetric property of $B_{\mu\nu}$, we can obtain

$$H_{\mu\nu\rho} + H_{\nu\mu\rho} = \partial_\mu(B_{\nu\rho} + B_{\rho\nu}) + \partial_\nu(B_{\rho\mu} + B_{\mu\rho}) + \partial_\rho(B_{\mu\nu} + B_{\nu\mu}) = 0 \tag{10.46}$$

Similarly, $H_{\mu\nu\rho} + H_{\rho\nu\mu} = 0$, $H_{\mu\nu\rho} + H_{\mu\rho\nu} = 0$. Therefore, $H_{\mu\nu\rho}$ is totally antisymmetric. Under the gauge transformation $\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$, we have

$$\delta H_{\mu\nu\rho} = \partial_\mu(\partial_\nu \epsilon_\rho - \partial_\rho \epsilon_\nu) + \partial_\nu(\partial_\rho \epsilon_\mu - \partial_\mu \epsilon_\rho) + \partial_\rho(\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu) = 0 \tag{10.47}$$

(b) It is easy to see that

$$\partial_\mu \epsilon'_\nu - \partial_\nu \epsilon'_\mu = \partial_\mu(\epsilon_\nu + \partial_\nu \lambda) - \partial_\nu(\epsilon_\mu + \partial_\mu \lambda) = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \tag{10.48}$$

(c) In the momentum space, $\epsilon'^\mu(p) = \epsilon^\mu + p^\mu \lambda(p)$ will generate the same gauge transformation as $\epsilon^\mu(p)$. Then, we have

$$\epsilon'^+(p) = \frac{1}{\sqrt{2}}\big(\epsilon^0 + p^0\lambda + \epsilon^1 + p^1\lambda\big) = 0 \Rightarrow \lambda = -\frac{\epsilon^+}{p^+} \tag{10.49}$$

(d)

10.7 (a) Recall that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, then we have

$$\begin{aligned} \delta\mathcal{L} &= -\frac{1}{4}\delta(F_{\mu\nu}F^{\mu\nu}) - m^2 A^\mu \partial_\mu \epsilon - bm\partial_\mu \phi \partial^\mu \epsilon - m\phi\partial^2\epsilon - bm^2(\partial \cdot A)\epsilon \\ &= -m^2 \partial_\mu(A^\mu \epsilon) - m\partial_\mu(\phi \partial^\mu \epsilon) \end{aligned} \tag{10.50}$$

where we have chosen $b = 1$. So the action will be invariant under the gauge transformation.

(b) The field equations are given by

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = 0 \Rightarrow m^2 A^\nu + \partial_\mu F^{\mu\nu} = 0 \tag{10.51}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 \Rightarrow \partial^2 \phi - m(\partial \cdot A) = 0 \tag{10.52}$$

(c) If we set $\epsilon = -\phi/m$, then $\phi' = \phi + \delta\phi = 0$. The second field equation becomes $\partial \cdot A = 0$.

(d) The simplified equations in the momentum space can be written as

$$(p^2 + m^2)A^\nu - (p \cdot A)p^\nu = 0, \; p \cdot A = 0 \Rightarrow (p^2 + m^2)A^\nu = 0 \tag{10.53}$$

If $p^2 \neq -m^2$, we can only have trivial solutions $A^\nu = 0$.

Chapter 11

The Relativistic Quantum Point Particle

■ Summary and Supplement

1. Light-cone point particle

$$\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \dot{x}^2, \quad S = \int_{\tau_i}^{\tau_f} L d\tau, \quad L = -m\sqrt{-\dot{x}^2} \tag{11.1}$$

$$x^+ = \frac{p^+}{m^2}\tau, \quad \dot{x}^2 = -\frac{1}{m^2}, \quad p_\mu = m^2\dot{x}_\mu, \quad p^2 + m^2 = 0 \tag{11.2}$$

$$p^- = \frac{1}{2p^+}(p^I p^I + m^2), \quad x^-(\tau) = x_0^- + \frac{p^-}{m^2}\tau, \quad x^I(\tau) = x_0^I + \frac{p^I}{m^2}\tau \tag{11.3}$$

2. Quantization of the point particle

$$\big(x^I, x_0^-, p^I, p^+\big), \quad \big(x^I(\tau), x_0^-(\tau), p^I(\tau), p^+(\tau)\big) \tag{11.4}$$

$$\big[x^I(\tau), p^J(\tau)\big] = i\eta^{IJ}, \quad \big[x_0^-(\tau), p^+(\tau)\big] = i\eta^{-+} = -i \tag{11.5}$$

$$H(\tau) = \frac{1}{2m^2}p^+(\tau)p^-(\tau) = \frac{1}{2m^2}\big[p^I(\tau)p^I(\tau) + m^2\big] \tag{11.6}$$

$$i\frac{\partial}{\partial\tau}\psi(\tau, p^+, \boldsymbol{p}_T) = \frac{1}{2m^2}\big(p^I p^I + m^2\big)\psi(\tau, p^+, \boldsymbol{p}_T) \tag{11.7}$$

$$\big|p^+, \boldsymbol{p}_T\big\rangle \leftrightarrow a^\dagger_{p^+,\boldsymbol{p}_T}\left|\Omega\right\rangle, \quad \psi(\tau, p^+, \boldsymbol{p}_T) \leftrightarrow \phi(\tau, p^+, \boldsymbol{p}_T) \tag{11.8}$$

3. Light-cone Lorentz generators

$$\delta x^\mu(\tau) = \epsilon^{\mu\nu}x_\nu(\tau), \quad M^{\mu\nu} = x^\mu(\tau)p^\nu(\tau) - x^\nu(\tau)p^\mu(\tau) \tag{11.9}$$

$$(M^{\mu\nu})^\dagger = M^{\mu\nu}, \quad \big[M^{\mu\nu}, x^\rho(\tau)\big] = i\eta^{\mu\rho}x^\nu(\tau) - i\eta^{\nu\rho}x^\mu(\tau) \tag{11.10}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\mu\sigma}M^{\rho\nu} - i\eta^{\nu\sigma}M^{\rho\mu} \tag{11.11}$$

$$M^{+-} = -\frac{1}{2}(x_0^- p^+ + p^+ x_0^-), \quad M^{-I} = x_0^- p^I - \frac{1}{2}(x_0^I p^- + p^- x_0^I) \tag{11.12}$$

■ Quick Calculations

11.1 For the state $\left|\Psi, t\right\rangle = e^{-iHt}\left|\Psi\right\rangle$, we can easily check that

$$i\frac{d}{dt}\left|\Psi, t\right\rangle = i(-iH)e^{-iHt}\left|\Psi\right\rangle = H\left|\Psi, t\right\rangle \tag{11.13}$$

11.2 For the corresponding Heisenberg operators, we have

$$\big[\alpha_1(t), \alpha_2(t)\big] = \big[e^{iHt}\alpha_1 e^{-iHt}, e^{iHt}\alpha_2 e^{-iHt}\big] = e^{iHt}\big[\alpha_1, \alpha_2\big]e^{-iHt} = \alpha_3(t) \tag{11.14}$$

11.3 Since $[x_0^-(\tau), p^I(\tau)] = 0$, then we can obtain

$$i\frac{dx_0^-(\tau)}{d\tau} = \frac{1}{2m^2}[x_0^-(\tau), p^I p^I + m^2] = 0 \tag{11.15}$$

11.4 For $\epsilon^- \neq 0$ and $\epsilon^+ = \epsilon^I = 0$, it is easy to check that

$$\delta x^-(\tau) = \big[i\epsilon_\rho p^\rho(\tau), x^-(\tau)\big] = \big[i\epsilon_+ p^+(\tau), x^-(\tau)\big] = -\epsilon_+ = \epsilon^- \tag{11.16}$$

11.5 Using the fact $[x_0^-, 1/p^+] = i/p^{+2}$, we can show that

$$\big[x_0^-, p^-\big] = \big[x_0^-, \frac{1}{2p^+}(p^I p^I + m^2)\big] = \frac{1}{2}\big[x_0^-, \frac{1}{p^+}\big](p^I p^I + m^2) = i\frac{p^I p^I + m^2}{2p^{+2}} = i\frac{p^-}{p^+} \tag{11.17}$$

11.6 According to Eq. (11.5), we have

$$\big[M^{\mu\nu}, x^\rho(\tau)\big] = x^\mu(\tau)\big[p^\nu(\tau), x^\rho(\tau)\big] - x^\nu(\tau)\big[p^\mu(\tau), x^\rho(\tau)\big] = i\eta^{\mu\rho}x^\nu(\tau) - i\eta^{\nu\rho}x^\mu(\tau) \tag{11.18}$$

11.7 We observe that: the first two terms are antisymmetric for the indices μ and ν, and the last two terms are also antisymmetric for the indices μ and ν.

11.8 By the definition, we can directly prove that

$$M^{+-} = x^+p^- - x^-p^+ = \frac{1}{2}(x^0+x^1)(p^0-p^1) - \frac{1}{2}(x^0-x^1)(p^0+p^1) = x^1p^0 - x^0p^1 = M^{10} \tag{11.19}$$

■ Solutions to the Problems

11.1 For the Heisenberg operator, we can prove its equation of motion

$$i\frac{d\xi(t)}{dt} = i\frac{d}{dt}e^{iHt}\xi e^{-iHt} = i(iHe^{iHt})\xi e^{-iHt} + ie^{iHt}\xi(-iHe^{-iHt}) = [\xi(t), H] \tag{11.20}$$

11.2 (a) According to the Schrödinger equation, we have

$$i\frac{d\,|\Psi, t\rangle}{dt} = i\frac{dU(t)}{dt}\,|\Psi\rangle = HU(t)\,|\Psi\rangle \Rightarrow i\frac{dU(t)}{dt} = HU(t) \tag{11.21}$$

(b) This result has been proven in Eq. (11.20).

(c) This result has been proven in Eq. (11.14).

11.3 Using the Hamilton's canonical equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{11.22}$$

we can prove the time evolution of an operator in the classical phase space:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial p}\frac{dp}{dt} + \frac{\partial v}{\partial q}\frac{dq}{dt} = \frac{\partial v}{\partial t} - \frac{\partial v}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial v}{\partial q}\frac{\partial H}{\partial p} = \frac{\partial v}{\partial t} + \{v, H\} \tag{11.23}$$

11.4 The variation δL can be derived as follows

$$\delta L = \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\delta\dot{x}^\mu = \frac{d}{d\tau}\Big(\frac{\partial L}{\partial \dot{x}^\mu}\Big)\lambda_\tau\partial_\tau X^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\partial_\tau(\lambda\partial_\tau X^\mu) = \partial_\tau(\lambda L) \tag{11.24}$$

And the associated charge is given by

$$\lambda(\tau)Q = \frac{\partial L}{\partial \dot{x}^\mu}\delta x^\mu(\tau) - \lambda(\tau)L = 0 \tag{11.25}$$

11.5 (a) $[M^{\mu\nu}, p^\rho] = [x^\mu p^\nu - x^\nu p^\mu, p^\rho] = [x^\mu, p^\rho]p^\nu - [x^\nu, p^\rho]p^\mu = i\eta^{\mu\rho}p^\nu - i\eta^{\nu\rho}p^\mu$.

(b) $[x^\mu p^\nu, x^\rho p^\sigma] = [x^\mu, x^\rho p^\sigma]p^\nu + x^\mu[p^\nu, x^\rho p^\sigma] = i\eta^{\mu\sigma}x^\rho p^\nu - i\eta^{\nu\rho}x^\mu p^\sigma$. Then, we have

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= [x^\mu p^\nu, x^\rho p^\sigma] - [x^\mu p^\nu, x^\sigma p^\rho] - [x^\nu p^\mu, x^\rho p^\sigma] + [x^\nu p^\mu, x^\sigma p^\rho] \\
&= (i\eta^{\mu\sigma}x^\rho p^\nu - i\eta^{\nu\rho}x^\mu p^\sigma) - (i\eta^{\mu\rho}x^\sigma p^\nu - i\eta^{\nu\sigma}x^\mu p^\rho) \\
&\quad - (i\eta^{\nu\sigma}x^\rho p^\mu - i\eta^{\mu\rho}x^\nu p^\sigma) + (i\eta^{\nu\rho}x^\sigma p^\mu - i\eta^{\mu\sigma}x^\nu p^\rho) \\
&= i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\mu\sigma}M^{\rho\nu} - i\eta^{\nu\sigma}M^{\rho\mu}
\end{aligned} \tag{11.26}$$

(c) In the light cone coordinates, we have $\eta^{+-} = \eta^{-+} = -1$, $\eta^{II} = 1$ and the others are zero.

11.6 (a) First, we calculate the value of $[x_0^I, p^-]$.

$$[x_0^I, p^-] = [x_0^I, \frac{1}{2p^+}(p^I p^I + m^2)] = \frac{p^I}{p^+}[x_0^I, p^I] = \frac{ip^I}{p^+} \tag{11.27}$$

Then, it is easy to obtain

$$M^{-I} = x_0^- p^I - \frac{1}{2}(x_0^I p^- + p^- x_0^I) = x_0^- p^I - x_0^I p^- + \frac{1}{2}[x_0^I, p^-] = x_0^- p^I - x_0^I p^- + \frac{ip^I}{2p^+} \tag{11.28}$$

(b) Now, we shall prove the light-cone gauge commutator

$$\begin{aligned}[M^{-I}, M^{-J}] &= [x_0^- p^I, \frac{ip^J}{2p^+}] - [x_0^- p^I, x_0^J p^-] - [x_0^I p^-, x_0^- p^J] + [x_0^I p^-, x_0^J p^-] + [\frac{ip^I}{2p^+}, x_0^- p^J] \\ &= -\frac{ip^I p^J}{2p^{+2}} - \frac{ix_0^J p^I p^-}{p^+} + \frac{ix_0^I p^J p^-}{p^+} + (\frac{ix_0^J p^I p^-}{p^+} - \frac{ix_0^I p^J p^-}{p^+}) + \frac{ip^I p^J}{2p^{+2}} = 0\end{aligned} \tag{11.29}$$

where we have used the following relations:

$$[x_0^I, p^-] = \frac{ip^I}{p^+}, \quad [x_0^-, \frac{1}{p^+}] = \frac{i}{p^{+2}}, \quad [x_0^-, p^-] = \frac{ip^-}{p^+} \tag{11.30}$$

11.7 (a) By the definition, we have

$$M^{+-} = -\frac{1}{2}(x^- - \frac{p^-}{m^2}\tau)p^+ - \frac{1}{2}p^+(x^- - \frac{p^-}{m^2}\tau) = \frac{p^+ p^-}{m^2}\tau - \frac{1}{2}(x^- p^+ + p^+ x^-) \tag{11.31}$$

(b)

Chapter 12

Relativistic Quantum Open Strings

■ Summary and Supplement

1. Light-cone Hamiltonian and commutators

$$(x^I(\sigma), x_0^-, \mathcal{P}^{\tau I}(\sigma), p^+), \quad \big(x^I(\tau,\sigma), x_0^-(\tau), \mathcal{P}^{\tau I}(\tau,\sigma), p^+(\tau)\big) \tag{12.1}$$

$$[X^I(\tau,\sigma), \mathcal{P}^{\tau J}(\tau,\sigma)] = i\eta^{IJ}\delta(\sigma-\sigma'), \quad [x_0^-(\tau), p^+(\tau)] = -i \tag{12.2}$$

$$X^+ = 2\alpha' p^+ \tau, \quad L_0^\perp = 2\alpha' p^+ p^-, \quad H = 2\alpha' p^+ \int_0^\pi d\sigma\, \mathcal{P}^{\tau -} \tag{12.3}$$

$$H(\tau) = \pi\alpha' \int_0^\pi d\sigma \left[\mathcal{P}^{\tau I}(\tau,\sigma)\mathcal{P}^{\tau I}(\tau,\sigma) + \frac{X^{I\prime}(\tau,\sigma)X^{I\prime}(\tau,\sigma)}{(2\pi\alpha')^2}\right] \tag{12.4}$$

2. Commutation relations for oscillators

$$\big[(\dot{X}^I \pm X^{I\prime})(\tau,\sigma), (\dot{X}^J \pm X^{J\prime})(\tau,\sigma')\big] = \pm 4\pi\alpha' i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma') \tag{12.5}$$

$$\big[(\dot{X}^I \pm X^{I\prime})(\tau,\sigma), (\dot{X}^J \mp X^{J\prime})(\tau,\sigma')\big] = 0, \quad \sigma,\sigma' \in [0,\pi] \tag{12.6}$$

$$\sum_{m',n'\in\mathbb{Z}} e^{-im'(\tau+\sigma)}e^{-in'(\tau+\sigma')}[\alpha^I_{m'}, \alpha^J_{n'}] = 2\pi i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma') \tag{12.7}$$

$$[\alpha^I_m, \alpha^J_{-n}] = m\delta_{mn}\eta^{IJ}, \quad \alpha^I_0 = \sqrt{2\alpha'}p^I \tag{12.8}$$

$$[x^I_0, \alpha^J_0] = \sqrt{2\alpha'}i\eta^{IJ}, \quad [x^I_0, \alpha^J_n] = 0, \quad n \neq 0 \tag{12.9}$$

$$\alpha^I_n = a^I_n\sqrt{n}, \quad \alpha^I_{-n} = a^{I\dagger}_n\sqrt{n}, \quad (\alpha^I_n)^\dagger = \alpha^I_{-n} \tag{12.10}$$

$$[a^I_m, a^{J\dagger}_n] = \delta_{mn}\eta^{IJ}, \quad [a^I_m, a^I_n] = [a^{I\dagger}_m, a^{J\dagger}_n] = 0 \tag{12.11}$$

3. Strings as harmonic oscillators

$$S = \int d\tau d\sigma\, \mathcal{L} = \frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma\, (\dot{X}^I\dot{X}^I - X^{I\prime}X^{I\prime}) \tag{12.12}$$

$$X^I(\tau,\sigma) = q^I(\tau) + 2\sqrt{\alpha'}\sum_{n=1}^\infty q^I_n(\tau)\frac{\cos n\sigma}{\sqrt{n}} \tag{12.13}$$

$$X^I(\tau,\sigma) = x^I_0 + 2\alpha' p^I\tau + i\sqrt{2\alpha'}\sum_{n=1}^\infty (a^I_n e^{-in\tau} - a^{I\dagger}_n e^{in\tau})\frac{\cos n\sigma}{\sqrt{n}} \tag{12.14}$$

4. Transverse Virasoro operators

$$\sqrt{2\alpha'}\alpha^-_n = \frac{1}{p^+}L^\perp_n, \quad L^\perp_n = \frac{1}{2}\sum_{p\in\mathbb{Z}}^\infty \alpha^I_{n-p}\alpha^I_p, \quad (L^\perp_n)^\dagger = L^\perp_{-n} \tag{12.15}$$

$$L^\perp_0 = \frac{1}{2}\alpha^I_0\alpha^I_0 + \sum_{p=1}^\infty \alpha^I_{-p}\alpha^I_p + \frac{1}{2}(D-2)\sum_{p=1}^\infty p \tag{12.16}$$

$$2\alpha' p^- = \frac{1}{p^+}(L_0^\perp + a), \quad M^2 = \frac{1}{\alpha'}(a + \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I) \tag{12.17}$$

$$a = \frac{1}{2}(D-2)\sum_{p=1}^{\infty} p, \quad \zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \zeta(-1) = -\frac{1}{12} \tag{12.18}$$

$$[L_m^\perp, \alpha_n^I] = -n\alpha_{m+n}^I, \quad [L_m^\perp, x_0^I] = -i\sqrt{2\alpha'}\alpha_m^I \tag{12.19}$$

$$[L_m^\perp, L_{-n}^\perp] = (m+n)L_{m-n}^\perp + \frac{D-2}{12}(m^3 - m)\delta_{mn} \tag{12.20}$$

$$[L_m^\perp, X^I(\tau,\sigma)] = \xi_m^\tau \dot{X}^I + \xi_m^\sigma X^{I\prime} \tag{12.21}$$

$$\xi_m^\tau(\tau,\sigma) = -ie^{im\tau}\cos m\sigma, \quad \xi_m^\sigma(\tau,\sigma) = e^{im\tau}\sin m\sigma \tag{12.22}$$

$$X^I(\tau + \epsilon\xi_m^\tau, \sigma + \epsilon\xi_m^\sigma) = X^I(\tau,\sigma) + \epsilon[L_m^\perp, X^I(\tau,\sigma)] \tag{12.23}$$

5. Lorentz generators

$$M^{\mu\nu} = \frac{1}{2\pi\alpha'}\int_0^\pi (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu)\, d\sigma \tag{12.24}$$

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\mu - \alpha_{-n}^\nu \alpha_n^\nu) \tag{12.25}$$

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\left[x_0^I(L_0^\perp + a) + (L_0^\perp + a)x_0^I\right] - \frac{i}{\sqrt{2\alpha'}p^+}\sum_{n=1}^{\infty}\frac{1}{n}(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp) \tag{12.26}$$

$$[M^{-I}, M^{-J}] = \frac{1}{\alpha' p^{+2}}\sum_{m=1}^{\infty}\left(\alpha_{-m}^I \alpha_m^J - \alpha_{-m}^J \alpha_m^I\right) \times \left[m\Big(1 - \frac{D-2}{24}\Big) + \frac{1}{m}\Big(\frac{D-2}{24} + a\Big)\right] \tag{12.27}$$

$$D = 26, \quad a = -1, \quad [M^{-I}, M^{-J}] = 0, \quad 2\alpha' p^- = \frac{1}{p^+}(L_0^\perp - 1), \quad H = L_0^\perp - 1 \tag{12.28}$$

6. Constructing the state space

$$a_n^I \left|p^+, \boldsymbol{p}_T\right\rangle = 0, \quad n \geq 1, \quad I = 2, \ldots, 25 \tag{12.29}$$

$$|\lambda\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{25}(a_n^{I\dagger})^{\lambda_{n.I}}\left|p^+, \boldsymbol{p}_T\right\rangle \tag{12.30}$$

$$N^\perp = \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I, \quad M^2 = \frac{1}{\alpha'}(-1 + N^\perp) \tag{12.31}$$

$$[N^\perp, a_n^{I\dagger}] = n a_n^{I\dagger}, \quad [N^\perp, a_n^I] = -n a_n^I \tag{12.32}$$

$$N^\perp |\lambda\rangle = N_\lambda^\perp |\lambda\rangle, \quad N_\lambda^\perp = \sum_{n=1}^{\infty}\sum_{I=2}^{25} n\lambda_{n,I} \tag{12.33}$$

$$\text{tachyons: } N^\perp = 0, \quad \alpha' M^2 = -1, \quad \text{number of states} = 1 \tag{12.34}$$

$$\text{photons: } N^\perp = 1, \quad \alpha' M^2 = 0, \quad \text{number of states} = D - 2 \tag{12.35}$$

$$\text{massive tensors: } N^\perp = 2, \quad \alpha' M^2 = 1, \quad \text{number of states} = (D-2)(D+1)/2 \tag{12.36}$$

7. Tachyons and D-brane decay

■ Quick Calculations

12.1 Since $(x_0^I)^\dagger = x_0^I$, $(\alpha_0^I)^\dagger = \alpha_0^I$, and the index n is summed over all integers except zero, we can see that $\big(X^I(\tau,\sigma)\big)^\dagger = X^I(\tau,\sigma)$.

12.2 Using the definition in Eq. (12.12), we can obtain

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma \left[\dot{q}^I(\tau)\dot{q}^I(\tau) + 4\alpha' \sum_{n=1}^{\infty} \dot{q}_n^I(\tau)\dot{q}_n^I(\tau)\frac{\cos^2 n\sigma}{n} - 4\alpha' \sum_{n=1}^{\infty} q_n^I(\tau)q_n^I(\tau) n \sin^2 n\sigma\right] \\ &= \frac{1}{4\pi\alpha'}\int d\tau \left[\pi\dot{q}^I(\tau)\dot{q}^I(\tau) + 4\alpha' \sum_{n=1}^{\infty} \dot{q}_n^I(\tau)\dot{q}_n^I(\tau)\frac{\pi}{2n} - 4\alpha' \sum_{n=1}^{\infty} q_n^I(\tau)q_n^I(\tau)\frac{n\pi}{2}\right] \\ &= \int d\tau \left[\frac{1}{4\alpha'}\dot{q}^I(\tau)\dot{q}^I(\tau) + \sum_{n=1}^{\infty}\left(\frac{1}{2n}\dot{q}_n^I(\tau)\dot{q}_n^I(\tau) - \frac{n}{2}q_n^I(\tau)q_n^I(\tau)\right)\right] \end{aligned} \tag{12.37}$$

12.3 Using the relations in Eq. (12.9), we can see that

$$[L_m^\perp, x_0^I] = \frac{1}{2}\sum_{p\in\mathbb{Z}}[\alpha_{m-p}^J\alpha_p^J, x_0^I] = [\alpha_0^I, x_0^J]\alpha_m^J = -i\sqrt{2\alpha'}\eta^{IJ}\alpha_m^J = -i\sqrt{2\alpha'}\alpha_m^I \tag{12.38}$$

12.4 For $p \neq 1$, $\alpha_{2-p}\alpha_p = \alpha_p\alpha_{2-p}$, therefore we have

$$L_2 = \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{2-p}\alpha_p = \frac{1}{2}\alpha_1\alpha_1 + (\alpha_0\alpha_2 + \alpha_{-1}\alpha_3 + \alpha_{-2}\alpha_4 + \cdots) \tag{12.39}$$

Similarly, for $p \neq -1$, $\alpha_{-2-p}\alpha_p = \alpha_p\alpha_{-2-p}$, therefore we have

$$L_{-2} = \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{-2-p}\alpha_p = \frac{1}{2}\alpha_{-1}\alpha_{-1} + (\alpha_{-2}\alpha_0 + \alpha_{-3}\alpha_1 + \alpha_{-4}\alpha_2 + \cdots) \tag{12.40}$$

12.5 For the oscillators including Lorentz indices, we can verify that

$$\begin{aligned} \frac{1}{4}[\alpha_1^I\alpha_1^I, \alpha_{-1}^J\alpha_{-1}^J] &= \frac{1}{2}(\alpha_1^I[\alpha_1^I, \alpha_{-1}^I]\alpha_{-1}^J + \alpha_{-1}^J[\alpha_1^I, \alpha_{-1}^I]\alpha_1^I) \\ &= \frac{1}{2}(\eta^{IJ}\alpha_1^I\alpha_{-1}^J + \eta^{IJ}\alpha_{-1}^J\alpha_1^I) = \frac{D-2}{2} + \alpha_{-1}^I\alpha_1^I \end{aligned} \tag{12.41}$$

12.6 It is easy to check the expressions. Note that the numbers n, $m-n$, p and $m-p$ that appear on the oscillator are all positive.

12.7 Suppose that $\sum_{n=1}^m n^2 = am^3 + bm^2 + cm$, then you can use the specila cases for $m = 1$, 2 , 3 to determine the values of a, b, c.

12.8 For the transformation generated by $i(L_m^\perp + L_{-m}^\perp)$, the parameters are given by

$$\xi^\tau = i(\xi_m^\tau + \xi_{-m}^\tau) = (e^{im\tau} + e^{-im\tau})\cos m\sigma = 2\cos m\tau \cos m\sigma \tag{12.42}$$

$$\xi^\sigma = i(\xi_m^\sigma + \xi_{-m}^\sigma) = i(e^{im\tau} - e^{-im\tau})\sin m\sigma = -2\sin m\tau \sin m\sigma \tag{12.43}$$

12.9 Since $M^{\mu\nu}$ is τ-independent, it suffices to pick up the τ-independent terms that arise in the products. Then, we have

$$\begin{aligned} M^{\mu\nu} &= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \left[\sqrt{2\alpha'}(x_0^\mu\alpha_0^\nu - x_0^\nu\alpha_0^\mu) + i2\alpha'\sum_{n\neq 0}\frac{1}{n}(\alpha_n^\mu\alpha_{-n}^\nu - \alpha_n^\nu\alpha_{-n}^\mu)\cos^2 n\sigma\right] \\ &= x_0^\mu p^\nu - x_0^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu\alpha_n^\mu - \alpha_{-n}^\nu\alpha_n^\nu) \end{aligned} \tag{12.44}$$

12.10 If the basis states $|\lambda\rangle$ and $|\lambda'\rangle$ are different, then we have

$$(\lambda', \lambda) = \delta^{IJ}\delta(p'^+ - p^+)\boldsymbol{\delta}(\boldsymbol{p}'_T - \boldsymbol{p}_T) = 0 \tag{12.45}$$

12.11 Substitute the expression for $|\Psi, t\rangle$ into the Schrödinger's equation satisfied by the general states, then the result will be obvious.

12.12 Using the Euler-Lagrange equation, we can directly obtain that

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 \Rightarrow -V'(\phi) + \partial_\mu \partial^\mu \phi = 0 \tag{12.46}$$

■ Solutions to the Problems

12.1

12.2 For the mode expansion of $X^I(\tau, \sigma)$, its derivatives are given by

$$\dot{X}^I(\tau, \sigma) = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha_n^I e^{-in\tau} \cos n\sigma, \quad X^{I\prime}(\tau, \sigma) = -i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha_n^I e^{-in\tau} \sin n\sigma \tag{12.47}$$

Using the commutator in Eq. (12.8), we can easily obtain

$$\begin{aligned} \left[X^{I\prime}(\tau, \sigma), X^{J\prime}(\tau, \sigma')\right] &= -2\alpha' \sum_{m \in \mathbb{Z}} \left[\alpha_m^I e^{-im\tau} \sin m\sigma, \sum_{n \in \mathbb{Z}} \alpha_n^J e^{-in\tau} \sin n\sigma'\right] \\ &= 2\alpha' \eta^{IJ} \sum_{m \in \mathbb{Z}} m \sin m\sigma \sin m\sigma' = 0 \end{aligned} \tag{12.48}$$

$$\begin{aligned} \left[\dot{X}^I(\tau, \sigma), \dot{X}^J(\tau, \sigma')\right] &= 2\alpha' \sum_{m \in \mathbb{Z}} \left[\alpha_m^I e^{-im\tau} \cos m\sigma, \sum_{n \in \mathbb{Z}} \alpha_n^J e^{-in\tau} \cos n\sigma'\right] \\ &= 2\alpha' \eta^{IJ} \sum_{m \in \mathbb{Z}} m \cos m\sigma \cos m\sigma' = 0 \end{aligned} \tag{12.49}$$

12.3 (a) Since α_0^I commutes with all other oscillators, it does not contribute to our calculations presented here. So we have

$$\begin{aligned} [X^I(\tau, \sigma), \mathcal{P}^{\tau J}(\tau, \sigma')] &= \left[x_0^I + i\sqrt{2\alpha'} \sum_{m \neq 0} \frac{1}{m} \alpha_m^I e^{-im\tau} \cos m\sigma, \frac{1}{\pi\sqrt{2\alpha'}} \sum_{n \in \mathbb{Z}} \alpha_n^J e^{-in\tau} \cos n\sigma'\right] \\ &= \frac{1}{\pi\sqrt{2\alpha'}} [x_0^I, \alpha_0^J] + \frac{i}{\pi} \eta^{IJ} \sum_{m \neq 0} \cos m\sigma \cos m\sigma' \\ &= i\eta^{IJ} \frac{1}{\pi} \sum_{m \in \mathbb{Z}} \cos m\sigma \cos m\sigma' \end{aligned} \tag{12.50}$$

(b) By the basic result of Fourier series, we can obtain

$$\int_0^\pi f(\sigma') \delta(\sigma - \sigma') \, d\sigma' = f(\sigma) = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \cos n\sigma \int_0^\pi f(\sigma') \cos n\sigma' \, d\sigma' \tag{12.51}$$

Then, it gives the following representation of the delta function

$$\delta(\sigma - \sigma') = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \cos n\sigma \cos n\sigma' \tag{12.52}$$

12.4 For $|x| < 1$, the series $\sum_{n=0}^\infty x^n = 1/(1-x)$ holds. Then, we have

$$\int_0^\infty dt \, \frac{t^s - 1}{e^t - 1} = \int_0^\infty dt \, \sum_{n=1}^\infty e^{-nt} = \sum_{n=1}^\infty \frac{1}{n^s} \int_0^\infty dx \, e^{-x} x^{s-1} = \Gamma(s)\zeta(s) \tag{12.53}$$

For the small t, the following expansion holds

$$\frac{1}{e^t - 1} \simeq \frac{1}{t(1 + \frac{t}{2} + \frac{t^2}{6})} = \frac{1}{t}\left[1 - \frac{t}{2} - \frac{t^2}{6} + \frac{t^2}{4} + \mathcal{O}(t^3)\right] = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^2) \tag{12.54}$$

And it is obvious that

$$\int_0^1 dt \, t^{s-1}\left(-\frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right) + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} = 0, \tag{12.55}$$

therefore we can rewrite Eq. (12.53) as

$$\Gamma(s)\zeta(s) = \int_0^1 dt\, t^{s-1}\Big(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\Big) + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^\infty dt\, \frac{1}{e^t-1} \tag{12.56}$$

Using the small t expansion, we can see that the function of the first integral has the order of $\mathcal{O}(t^{s+1})$. For $\Re(s) > -2$, the integral on $[0,1]$ will always converge. So the right-hand side above is well defined. Recall the pole struction of $\Gamma(s)$: $\text{Res}[\Gamma(s), -n] = (-1)^n/n!$, we have

$$\text{Res}[\Gamma(s),0]\zeta(0) = -\frac{1}{2} \Rightarrow \zeta(0) = -\frac{1}{2} \quad \text{Res}[\Gamma(s),-1]\zeta(-1) = \frac{1}{12} \Rightarrow \zeta(-1) = -\frac{1}{12} \tag{12.57}$$

We can also obtain the result from the following formula:

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s) \tag{12.58}$$

12.5 For $m+n \neq 0$, $[L^\perp_m, L^\perp_n] = (m-n)L^\perp_{m+n}$. Then, we can check that

$$[L^\perp_m, L^\perp_n] = (m-n)L^\perp_{m+n} = -(n-m)L^\perp_{n+m} = -[L^\perp_n, L^\perp_m] \tag{12.59}$$

$$\begin{aligned} &[L^\perp_m,[L^\perp_n,L^\perp_k]] + [L^\perp_n,[L^\perp_k,L^\perp_m]] + [L^\perp_k,[L^\perp_m,L^\perp_n]] \\ =&[L^\perp_m,(n-k)L^\perp_{n+k}] + [L^\perp_n,(k-m)L^\perp_{k+m}] + [L^\perp_k,(m-n)L^\perp_{m+n}] \\ =&\big[(n-k)(m-n-k) + (k-m)(n-k-m) + (m-n)(k-m-n)\big]L^\perp_{m+n+k} \\ =&0 \end{aligned} \tag{12.60}$$

For the Virasoro operators built with just one type of oscillator, we have

$$[L_m, L_n] = -\big[(n-m)L_{n+m} + \frac{1}{12}(n^3-n)\delta_{n+m,0}\big] = -[L_n, L_m] \tag{12.61}$$

$$\begin{aligned} &[L_m,[L_n,L_k]] + [L_n,[L_k,L_m]] + [L_k,[L_m,L_n]] \\ =&[L_m,(n-k)L_{n+k}] + [L_n,(k-m)L_{k+m}] + [L_k,(m-n)L_{m+n}] \\ =&\big[(n-k)(m-n-k) + (k-m)(n-k-m) + (m-n)(k-m-n)\big]L_{m+n+k} \\ &+ \frac{1}{12}\big[(n-k)(m^3-m) + (k-m)(n^3-n) + (m-n)(k^3-k)\big]\delta_{m+n+k,0} \\ =&0 \end{aligned} \tag{12.62}$$

12.6 (a) $A(-m) = -A(m)$, $A(0) = 0$.

(b) Now, we consider the Jacobi identity for L_m, L_n and L_k with $m+n+k=0$.

$$\begin{aligned} &[L_m,[L_n,L_k]] + [L_n,[L_k,L_m]] + [L_k,[L_m,L_n]] \\ =&[L_m,(n-k)L_{n+k}] + [L_n,(k-m)L_{k+m}] + [L_k,(m-n)L_{m+n}] \\ =&\big[(n-k)(m-n-k) + (k-m)(n-k-m) + (m-n)(k-m-n)\big]L_0 \\ &+ \big[(n-k)A(m) + (k-m)A(n) + (m-n)A(k)\big] \\ =&(n-k)A(m) + (k-m)A(n) + (m-n)A(k) \end{aligned} \tag{12.63}$$

(c) From the identity above, we can obtain the following difference equation:

$$(m-n)A(m+n) = (m+2n)A(m) - (2m+n)A(n) \tag{12.64}$$

12.7 (a) If $L_1|\lambda\rangle = 0$ and $L_2|\lambda\rangle = 0$, then we have $[L_2, L_1]|\lambda\rangle = L_3|\lambda\rangle = 0$, $[L_3, L_1]|\lambda\rangle = 2L_4|\lambda\rangle = 0$, and so on. Therefore, the state is annihilated by all L_n with $n \geq 1$.

(b) $[L_1, L_0] = L_1$, $[L_0, L_{-1}] = L_{-1}$, $[L_1, L_{-1}] = 2L_0$. They form a subalgebra of the Virasoro algebra. There are no central terms here.

12.8 (a) The combination $L^\perp_m - L^\perp_{-m}$ reparametrizes the σ coordinate of the string while keeping $\tau = 0$. They form a subalgebra of the Virasoro algebra. For $m \neq \pm n$, we have

$$[L^\perp_m - L^\perp_{-m}, L^\perp_n - L^\perp_{-n}] = (m-n)(L^\perp_{m+n} - L^\perp_{-m-n}) - (m+n)(L^\perp_{m-n} - L^\perp_{n-m}) \tag{12.65}$$

(b)

12.9 (a) It is easy to verify that

$$\dot{\xi}^{\tau}_{m} = me^{im\tau}\cos m\sigma = \xi^{\sigma\prime}_{m}, \quad \dot{\xi}^{\sigma}_{m} = ime^{im\tau}\sin m\sigma = \xi^{\tau\prime}_{m} \tag{12.66}$$

(b) For the change of coordinates, we have

$$\partial_\tau X = (1+\epsilon\partial_\tau\xi^\tau)\partial_{\tau'}X, \quad \partial_\sigma X = (1+\epsilon\partial_\sigma\xi^\sigma)\partial_{\sigma'}X \tag{12.67}$$

$$\partial_\tau X\cdot\partial_\sigma X = (1+\epsilon\partial_\sigma\xi^\sigma)^2\partial_{\tau'}X\cdot\partial_{\sigma'}X = 0 \Rightarrow \partial_{\tau'}X\cdot\partial_{\sigma'}X = 0 \tag{12.68}$$

$$(\partial_\tau X)^2 + (\partial_\sigma X)^2 = (1+\epsilon\partial_\sigma\xi^\sigma)^2\big[(\partial_{\tau'}X)^2 + (\partial_{\sigma'}X)^2\big] = 0 \Rightarrow (\partial_{\tau'}X)^2 + (\partial_{\sigma'}X)^2 = 0 \tag{12.69}$$

12.10 (a) If the orientation of this second string is the direction of decreasing σ, then it equals the first one.

(b)

12.11 (b) The critical points are given by $V'(\phi) = 0$.

$$V_1'(\phi) = \frac{1}{\alpha'\phi_0}\phi(\phi-\phi_0) = 0 \Rightarrow \phi_c = 0,\ \phi_0 \tag{12.70}$$

$$V_2'(\phi) = -\frac{1}{2\alpha'}|\phi|\Big(1+\ln\frac{\phi^2}{\phi_0^2}\Big) = 0 \Rightarrow \phi_c = 0,\ \pm\frac{\phi_0}{\sqrt{e}} \tag{12.71}$$

$$V_3'(\phi) = \frac{1}{2\alpha'\phi_0^2}\phi(\phi^2-\phi_0^2) = 0 \Rightarrow \phi_c = 0,\ \pm\phi_0 \tag{12.72}$$

(c) The mass of the scalar particle for the critical point $\bar{\phi}$ is given by $m(\bar{\phi}) = V''(\bar{\phi})$.

$$V_1''(\phi) = \frac{1}{\alpha'\phi_0}(2\phi-\phi_0) \Rightarrow m(0) = -\frac{1}{\alpha'},\ m(\phi_0) = \frac{1}{\alpha'} \tag{12.73}$$

$$V_2''(\phi) = -\frac{1}{2\alpha'}\Big(2+\ln\frac{\phi^2}{\phi_0^2}\Big) \Rightarrow m(0) = \infty,\ m\Big(\pm\frac{\phi_0}{\sqrt{e}}\Big) = -\frac{1}{2\alpha'} \tag{12.74}$$

$$V_3''(\phi) = \frac{1}{2\alpha'\phi_0^2}(3\phi^2-\phi_0^2) \Rightarrow m(0) = -\frac{1}{2\alpha'},\ m(\pm\phi_0) = \frac{1}{\alpha'} \tag{12.75}$$

Chapter 13

Relativistic Quantum Closed Strings

■ Summary and Supplement

1. Mode expansions and commutation relations

$$X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma), \quad X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi) \tag{13.1}$$

$$X_L^\mu(u) = \frac{1}{2}x_0^{L\mu} + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^\mu u + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu e^{-inu} \tag{13.2}$$

$$X_R^\mu(v) = \frac{1}{2}x_0^{R\mu} + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu v + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-inv} \tag{13.3}$$

$$\bar{\alpha}_0^\mu = \alpha_0^\mu, \qquad \alpha_0^\mu = \frac{\alpha'}{2}p^\mu, \qquad x_0^{L\mu} = x_0^{R\mu} = x_0^\mu \tag{13.4}$$

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}e^{-in\tau}(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}) \tag{13.5}$$

$$\dot{X}^\mu + X^{\mu\prime} = 2X_L^{\mu\prime}(\tau+\sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\bar{\alpha}_n^\mu e^{-in(\tau+\sigma)} \tag{13.6}$$

$$\dot{X}^\mu - X^{\mu\prime} = 2X_R^{\mu\prime}(\tau-\sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in(\tau-\sigma)} \tag{13.7}$$

$$[\bar{\alpha}_m^I, \bar{\alpha}_{-n}^J] = m\delta_{mn}\eta^{IJ}, \quad [\alpha_m^I, \alpha_{-n}^J] = m\delta_{mn}\eta^{IJ}, \quad [\alpha_m^I, \bar{\alpha}_n^J] = 0 \tag{13.8}$$

$$[\bar{a}_m^I, \bar{a}_n^{J\dagger}] = \delta_{mn}\eta^{IJ}, \quad [a_m^I, a_n^J] = \delta_{mn}\eta^{IJ}, \quad [x_0^I, p^J] = i\eta^{IJ} \tag{13.9}$$

$$X^+ = \alpha' p^+\tau, \qquad \partial_\tau = \alpha' p^+\partial_{X^+}, \qquad H = \alpha' p^+ p^- \tag{13.10}$$

2. Closed string Virasoro operators

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{2\alpha' p^+}(\dot{X}^- \pm X^{-\prime})^2 \tag{13.11}$$

$$(\dot{X}^- + X^{-\prime})^2 = 4\alpha'\sum_{n\in\mathbb{Z}}\bar{L}_n^\perp e^{-in(\tau+\sigma)}, \quad (\dot{X}^- - X^{-\prime})^2 = 4\alpha'\sum_{n\in\mathbb{Z}}L_n^\perp e^{-in(\tau-\sigma)} \tag{13.12}$$

$$\sqrt{2\alpha'}\bar{\alpha}_n^- = \frac{2}{p^+}(\bar{L}_n^\perp - 1), \quad \sqrt{2\alpha'}\alpha_n^- = \frac{2}{p^+}(L_n^\perp - 1) \tag{13.13}$$

$$\bar{L}_0^\perp = \frac{\alpha'}{4}p^I p^I + \bar{N}^\perp, \quad L_0^\perp = \frac{\alpha'}{4}p^I p^I + N^\perp, \quad \bar{L}_0^\perp = L_0^\perp \tag{13.14}$$

$$\bar{N}^\perp = \sum_{n=1}^\infty n\bar{a}_n^{I\dagger}\bar{a}_n^I, \quad N^\perp = \sum_{n=1}^\infty n a_n^{I\dagger}a_n^I, \quad \bar{N}^\perp = N^\perp \tag{13.15}$$

$$\frac{1}{p^+}(L_0^\perp + \bar{L}_0^\perp - 2) = \alpha' p^-, \quad M^2 = -p^2 = \frac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2) \tag{13.16}$$

3. Closed string states space

$$R_{IJ} = \hat{S}_{IJ} + A_{IJ} + S'\delta_{IJ}, \quad \text{tr}\hat{S}_{IJ} = 0, \quad S' = \frac{S}{D-2} \tag{13.17}$$

$$\text{Graviton fields:} \quad \sum_{I,J} \hat{S}_{IJ} a_1^{I\dagger} \bar{a}_1^{J\dagger} \left|p^+, \boldsymbol{p}_T\right\rangle \tag{13.18}$$

$$\text{Kalb-Ramond fields:} \quad \sum_{I,J} A_{IJ} a_1^{I\dagger} \bar{a}_1^{J\dagger} \left|p^+, \boldsymbol{p}_T\right\rangle \tag{13.19}$$

$$\text{Dilaton fields:} \quad S' a_1^{I\dagger} \bar{a}_1^{J\dagger} \left|p^+, \boldsymbol{p}_T\right\rangle \tag{13.20}$$

4. A brief look at superstring theories

$$G^{(10)} \sim g^2(\alpha')^4, \quad G \sim g^2\alpha', \quad g_o^2 \sim g, \quad g \sim e^{\phi} \tag{13.21}$$

$$\psi_1^I(\tau,\sigma) = \Psi_1^I(\tau-\sigma), \quad \psi_2^I(\tau,\sigma) = \Psi_2^I(\tau+\sigma) \tag{13.22}$$

$$\psi_1^I(\tau,0) = \psi_2^I(\tau,0), \quad \psi_1^I(\tau,\pi) = \pm\psi_2^I(\tau,\pi) \tag{13.23}$$

$$\text{Ramond boundary condition:} \quad \Psi^I(\tau,\pi) = +\Psi^I(\tau,-\pi) \tag{13.24}$$

$$\text{Neveu-Schwarz boundary condition:} \quad \Psi^I(\tau,\pi) = -\Psi^I(\tau,-\pi) \tag{13.25}$$

$$\Psi^I(\tau,\sigma) = \sum_{r\in\mathbb{Z}+1/2} b_r^I e^{-ir(\tau-\sigma)}, \quad b_{-r}^I b_{-r}^I = -b_{-r}^I b_{-r}^I = 0 \tag{13.26}$$

$$\text{NS sector:} \quad \prod_{I=2}^{9}\prod_{n=1}^{\infty}(\alpha_{-n}^I)^{\lambda_{n,I}} \prod_{J=2}^{9}\prod_{r=\frac{1}{2},\frac{3}{2},\cdots}(b_{-r}^J)^{\rho_{r,J}} \left|\text{NS}\right\rangle \otimes \left|p^+, \boldsymbol{P}_T\right\rangle \tag{13.27}$$

$$\Psi^I(\tau,\sigma) = \sum_{n\in\mathbb{Z}} d_n^I e^{-ir(\tau-\sigma)}, \quad \text{ground states: } \left|\text{R}^A\right\rangle, \ A = 1,\ldots,16 \tag{13.28}$$

$$\text{R sector:} \quad \prod_{I=2}^{9}\prod_{n=1}^{\infty}(\alpha_{-n}^I)^{\lambda_{n,I}} \prod_{J=2}^{9}\prod_{m=1}^{\infty}(d_{-m}^J)^{\rho_{m,J}} \left|\text{R}^A\right\rangle \otimes \left|p^+, \boldsymbol{P}_T\right\rangle \tag{13.29}$$

$$\text{NS Sector: } \alpha' M^2 = -\frac{1}{2} + N^\perp, \quad \text{R Sector: } \alpha' M^2 = N^\perp \tag{13.30}$$

$$\text{bosonic states: } b_{-1/2}^I \left|\text{NS}\right\rangle \otimes \left|p^+, \boldsymbol{p}_T\right\rangle, \quad \text{fermionic states: } \left|\text{R}_1^a\right\rangle \otimes \left|p^+, \boldsymbol{p}_T\right\rangle \tag{13.31}$$

$$\text{NS-NS massless fields: } g_{\mu\nu}, \ B_{\mu\nu}, \ \phi, \quad b_{-1/2}^I \bar{b}_{-1/2}^J \left|\text{NS}\right\rangle_L \otimes \left|\text{NS}\right\rangle_R \otimes \left|p^+, \boldsymbol{p}_T\right\rangle \tag{13.32}$$

$$\text{Type IIA}, \quad \text{Type IIB}, \quad E_8\times E_8 \text{ heterotic}, \quad SO(32) \text{ heterotic}, \quad \text{Type I} \tag{13.33}$$

■ Quick Calculations

13.1 By the definition of $\bar{L}_m^\perp$ and $L_m^\perp$, we can obtain

$$[\bar{L}_m^\perp, x_0^I] = \frac{1}{2}\sum_{p\in\mathbb{Z}}[\bar{\alpha}_p^J\bar{\alpha}_{m-p}^J, x_0^I] = [\bar{\alpha}_0^J, x_0^I]\bar{\alpha}_m^J = -i\sqrt{\frac{\alpha'}{2}}\eta^{IJ}\bar{\alpha}_m^J = -i\sqrt{\frac{\alpha'}{2}}\bar{\alpha}_m^I \tag{13.34}$$

$$[L_m^\perp, x_0^I] = \frac{1}{2}\sum_{p\in\mathbb{Z}}[\alpha_p^J\alpha_{m-p}^J, x_0^I] = [\alpha_0^J, x_0^I]\alpha_m^J = -i\sqrt{\frac{\alpha'}{2}}\eta^{IJ}\alpha_m^J = -i\sqrt{\frac{\alpha'}{2}}\alpha_m^I \tag{13.35}$$

13.2 First, we point out that the following relations hold

$$[\bar{L}_m^\perp, \bar{\alpha}_n^I] = -n\bar{\alpha}_{m+n}^I, \quad [L_m^\perp, \alpha_n^I] = -n\alpha_{m+n}^I, \quad [L_m^\perp, \bar{\alpha}_n^I] = [\bar{L}_m^\perp, \alpha_n^I] = 0 \tag{13.36}$$

Then, using Eq. (13.5), we can prove that

$$\begin{aligned}
[\bar{L}_0^\perp, X^I(\tau,\sigma)] &= [\bar{L}_0^\perp, x_0^I] + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}e^{-in\tau}[\bar{L}_0^\perp, \bar{\alpha}_n^I]e^{-in\sigma} \\
&= -i\sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^I - i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\bar{\alpha}_n^I e^{-in(\tau+\sigma)} \\
&= -\frac{i}{2}(\dot{X}^\mu + X^{\mu\prime})
\end{aligned} \tag{13.37}$$

$$\begin{aligned}[L_0^\perp, X^I(\tau,\sigma)] &= [L_0^\perp, x_0^I] + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}e^{-in\tau}[L_0^\perp, \alpha_n^I]e^{in\sigma} \\ &= -i\sqrt{\frac{\alpha'}{2}}\alpha_0^I - i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\alpha_n^I e^{-in(\tau-\sigma)} \\ &= -\frac{i}{2}(\dot{X}^\mu - X^{\mu\prime})\end{aligned} \tag{13.38}$$

13.3 The 16 ground states are listed as follows:

$$|\mathrm{R}_1^a\rangle: \quad |0\rangle\,,\ \xi_2\xi_1|0\rangle\,,\ \xi_3\xi_1|0\rangle\,,\ \xi_4\xi_1|0\rangle\,,\ \xi_3\xi_2|0\rangle\,,\ \xi_4\xi_2|0\rangle\,,\ \xi_4\xi_3|0\rangle\,,\ \xi_4\xi_3\xi_2\xi_1|0\rangle \tag{13.39}$$
$$|\mathrm{R}_2^a\rangle: \quad \xi_1|0\rangle\,,\ \xi_2|0\rangle\,,\ \xi_3|0\rangle\,,\ \xi_4|0\rangle\,,\ \xi_3\xi_2\xi_1|0\rangle\,,\ \xi_4\xi_2\xi_1|0\rangle\,,\ \xi_4\xi_3\xi_1|0\rangle\,,\ \xi_4\xi_3\xi_2|0\rangle \tag{13.40}$$

We can see that the eight ground states $|\mathrm{R}_1^a\rangle$ have an even number of fermionic operators and the other eight states $|\mathrm{R}_2^a\rangle$ have an odd number of fermionic operators.

13.4 From Eq. (13.26), we can conlcude that all states in the turncated NS sector have half-integer $N^\perp$ eigenvalues and integrally valued $\alpha' M^2$.

13.5 The numbers of graviton, Kalb-Ramond, and dilaton states in ten dimensions are 35, 28, and 1. Add these numbers up and we just get 64.

■ Solutions to the Problems

13.1 (a)

(b) Using Eq. (13.7), we have

$$\begin{aligned}[(\dot{X}^I - X^{I\prime})(\tau,\sigma), (\dot{X}^J - X^{J\prime})(\tau,\sigma')] &= 2\alpha'\sum_{m\in\mathbb{Z}} e^{-im(\tau-\sigma)}[\alpha_m^I, \sum_{n\in\mathbb{Z}}\alpha_n^J e^{-in(\tau-\sigma')} \\ &= 2\alpha'\sum_{m\in\mathbb{Z}} m\eta^{IJ}e^{im(\sigma-\sigma')} \\ &= -4\pi\alpha' i\eta^{IJ}\frac{d}{d\sigma}\delta(\sigma-\sigma')\end{aligned} \tag{13.41}$$

Then, it is obvious that the following holds

$$\delta(\sigma-\sigma') = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{im(\sigma-\sigma')} \tag{13.42}$$

13.2

13.3 (a) Define $f(\sigma_0) = e^{-iP\sigma_0}X^I(\tau,\sigma)e^{iP\sigma_0}$, then we have

$$\frac{df}{d\sigma_0} = e^{-iP\sigma_0}\Big(\frac{\partial X^I}{\partial\sigma_0}\Big)e^{iP\sigma_0}, \quad \frac{d^2 f}{d\sigma_0^2} = e^{-iP\sigma_0}\Big(\frac{\partial^2 X^I}{\partial\sigma_0^2}\Big)e^{iP\sigma_0}, \quad \ldots \tag{13.43}$$

By the Taylor's theorem, we can obtain

$$f(\sigma_0) = f(0) + \sum_{n=0}^{\infty}\frac{\sigma_0^n}{n!}\Big(\frac{d^n f}{d\sigma_0^n}\Big)_{\sigma_0=0} = \sum_{n=0}^{\infty}\frac{\sigma_0^n}{n!}\Big(\frac{\partial^2 X^I}{\partial\sigma_0^2}\Big) = X^I(\tau,\sigma+\sigma_0) \tag{13.44}$$

(b) The relaion holds for the following reasons

$$\frac{\partial}{\partial\tau}\Big[e^{-iP\sigma_0}X^I(\tau,\sigma)e^{iP\sigma_0}\Big] = e^{-iP\sigma_0}\dot{X}^I(\tau,\sigma)e^{iP\sigma_0} \tag{13.45}$$

$$\frac{\partial}{\partial\sigma}\Big[e^{-iP\sigma_0}X^I(\tau,\sigma)e^{iP\sigma_0}\Big] = e^{-iP\sigma_0}X^{I\prime}(\tau,\sigma)e^{iP\sigma_0} \tag{13.46}$$

(c) Using Eqs. (13.6) and (13.7), we can obtain

$$\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} e^{-iP\sigma_0}\bar{\alpha}_n^I e^{iP\sigma_0}e^{-in(\tau+\sigma)} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\bar{\alpha}_n^I(\tau,\sigma)e^{iP\sigma_0}e^{-in(\tau+\sigma+\sigma_0)} \tag{13.47}$$

$$\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} e^{-iP\sigma_0}\alpha_n^I e^{iP\sigma_0} e^{-in(\tau-\sigma)} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}} \alpha_n^I(\tau,\sigma) e^{iP\sigma_0} e^{-in(\tau-\sigma-\sigma_0)} \tag{13.48}$$

$$e^{-iP\sigma_0}\bar{\alpha}_n^I e^{iP\sigma_0} = \bar{\alpha}_n^I e^{-in\sigma_0}, \quad e^{-iP\sigma_0}\alpha_n^I e^{iP\sigma_0} = \alpha_n^I e^{in\sigma_0} \tag{13.49}$$

(d) Using the reults above, we have

$$e^{-iP\sigma_0}|U\rangle = (e^{-iP\sigma_0}\alpha_{-m}^I e^{iP\sigma_0})(e^{iP\sigma_0}\bar{\alpha}_{-n}^I e^{iP\sigma_0})|U\rangle = \alpha_{-m}^I \bar{\alpha}_{-n}^I e^{-i(m-n)\sigma_0}|U\rangle \tag{13.50}$$

The invariance of $|U\rangle$ under σ_0 transformation requires $\sigma_0 = 2k\pi$.

13.4 (a) Using Eq. (13.12), we can obtain

$$\int_0^{2\pi} d\sigma \dot{X}^I X^{I\prime} = \alpha' \sum_{n\in\mathbb{Z}} \int_0^{2\pi} d\sigma\, e^{-in\tau}(\bar{L}_n^\perp - e^{-in\sigma} - L_n^\perp e^{in\sigma}) = 2\pi\alpha'(\bar{L}_0^\perp - L_0^\perp) \tag{13.51}$$

13.5

13.6

13.7 (a) $b^{i_1}b^{i_2}$: $8\times7+1=57$. $b^{i_1}b^{i_2}b^{i_3}$: $8\times7\times6+1=337$. $b^{i_1}b^{i_2}b^{i_3}b^{i_4}$: $8\times7\times6\times5+1=1681$.

Part II

DEVELOPMENTS

Chapter 14

D-branes and Gauge Fields

■ Summary and Supplement

1. Quantizing open strings on Dp-branes

$$\text{DD:} \quad X^a(\tau,\sigma)\big|_{\sigma=0} = X^a(\tau,\sigma)\big|_{\sigma=\pi} = \bar{x}^a, \quad a = p+1,\ldots,d \tag{14.1}$$

$$\text{NN:} \quad X^{m\prime}(\tau,\sigma)\big|_{\sigma=0} = X^{m\prime}(\tau,\sigma)\big|_{\sigma=\pi} = 0, \quad m = 0,1,\ldots,p \tag{14.2}$$

(14.3)

(14.4)

(14.5)

2.

(14.6)

(14.7)

3.

(14.8)

(14.9)

■ Quick Calculations

14.1

14.2

14.3

■ Solutions to the Problems

14.1

14.2

14.3

Appendices

Appendix A

Elements of E_8

A.1 Introduction

Source from: http://en.wikipedia.org/wiki/E8_(mathematics)

In mathematics, E_8 is the name given to several closely related exceptional simple Lie groups and Lie algebras of dimension 248; the same notation is sometimes used for their root lattice, which has rank 8. The designation E_8 comes from Killing and Cartan's classification of the complex simple Lie algebras, which fall into four infinite families labeled A_n, B_n, C_n, D_n, and five exceptional cases labeled E_6, E_7, E_8, F_4, and G_2. The E_8 algebra is the largest and most complicated of these exceptional cases.

The E_8 Lie group has applications in theoretical physics, in particular in string theory and supergravity. The group $E_8 \times E_8$ serves as the gauge group of one of the two types of heterotic string and is one of two anomaly-free gauge groups that can be coupled to the $\mathcal{N} = 1$ supergravity in 10 dimensions. E_8 is the U-duality group of supergravity on an eight-torus (in its split form). One way to incorporate the standard model of particle physics into heterotic string theory is the symmetry breaking of E_8 to its maximal subalgebra $SU(3) \times E_6$. In 1982, Michael Freedman used the E_8 lattice to construct an example of a topological 4-manifold, the E_8 manifold, which has no smooth structure. In February 2008, Garret Lisi published a particle physics theory based on the E_8 Lie group.

Appendix B

Modular Forms

B.1 Introduction

Source from: http://en.wikipedia.org/wiki/Modular_form

In mathematics, a modular form is a (complex) analytic function on the upper half-plane satisfying a certain kind of functional equation and growth condition. The theory of modular forms therefore belongs to complex analysis but the main importance of the theory has traditionally been in its connections with number theory. Modular forms appear in other areas, such as algebraic topology and string theory. A modular function is a modular form of weight 0: it is invariant under the modular group, instead of transforming in a prescribed way, and is thus a function on the modular region (rather than a section of a line bundle). Modular form theory is a special case of the more general theory of automorphic forms, and therefore can now be seen as just the most concrete part of a rich theory of discrete groups.

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